Steady MHD Natural Convection Heat and Mass Transfer Flow above a Vertical Porous Surface with Thermal Diffusion and Inclined Magnetic Field

by

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Declaration

This is to certify that the thesis work entitled "***Steady MHD Natural Convection Heat and Mass Transfer Flow above a Vertical Porous Surface with Thermal Diffusion and Inclined Magnetic Field***" has been carried out by ***Md. Sohel Rana*** in the Department of *Mathematics*, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Dedication

TO

My Parents

&

My beloved niece

Arisha Mahmud Roha

Who have chosen under privileged life to continue my smile.

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Abstract

In this study the effect thermal diffusion and inclined magnetic field on the steady laminar natural convection heat and mass transfer flow of viscous incompressible MHD electrically conducting fluid past a vertical porous surface is considered under the influence of induced magnetic field. The governing non-dimensional equations relevant to the problem, containing the partial differential equations, are transformed by usual similarity transformations into a system of coupled non-linear ordinary differential equations and have been solved by using the perturbation technique. On introducing the non-dimensional concept and applying usual Boussinesq's approximation, the solutions for velocity fields, temperature distribution, induced magnetic fields and mass concentration are obtained up to the second order approximations for different selected values of the established dimensionless parameters and numbers. The influences of these various establish parameters and numbers on the velocity and temperature fields, induced magnetic field and mass concentration are exhibited under certain assumptions and are studied graphically. Further, the effects of these dimensionless parameters on the coefficients of skin friction and rate of heat and mass transfers are also studied in tabular form in the present analysis. The effect of different angle of inclinations of the externally applied uniform magnetic field on the field variables have been investigated for the present problem. It is observed that with other useful associated parameters, the thermal diffusion and the angle of inclination of the applied magnetic field have a retarding influence on the fluid velocity, induced magnetic field and mass concentration as well. It is also found that the dimensionless Prandtl number, Grashof number, Modified Grashof number, magnetic parameter and suction parameter have their own dependency on the concerned independent field variables like the velocity, temperature, concentration and induced magnetic fields as well as on other physical parameters of interests like local skin-friction coefficient (C_f), Nusselt number (N_u) and Sherwood number (S_h).

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Nomenclature

x, y	Cartesian coordinates
T	time
f_w	Suction parameter
U	Fluid velocity in the x direction
V	Fluid velocity in the y direction
K	Thermal conductivity
μ	Kinematic viscosity
η	Similarity variable
υ	Coefficient of kinematics viscosity
ψ	Stream functions
ρ	Fluid density
θ	Dimensionless temperature
γ	Boundary layer thickness
τ_w	Skin friction coefficient
φ	Velocity potential
P	Pressure
$f(\eta)$	Non-dimensional stream function
β	Coefficient of thermal expansion
q_w	Heat transfer coefficient
u_e	External velocity
L_c	Characteristic length
T	Fluid temperature
C	Concentration of species
μ_e	Magnetic permeability

H_0	Constant induced magnetic field
B_0	Applied magnetic field magnitude
D_m	Chemical molecular
D_T	Thermal diffusivity
C_w	Concentration of species at the wall
k_T	Thermal diffusion ratio
U_0	Uniform velocity
V_0	Non-zero suction velocity
G	Acceleration due to gravity
g_x	*x* component body force
R_e	Reynolds number
M	Magnetic Parameter
C_p	Specific heat capacity
G_r	Grashof number
G_m	Modified Grashof number
F_r	Froude number
P_r	Prandtl number
E_c	Eckert number
S_c	Schmidt number
S_0	Soret number
C_f	Skin-friction coefficient
N_u	Nusselt number
S_h	Sherwood number

INTRODUCTION

The thermal and mass transfer caused by the convection process takes place due to the buoyancy effects owing to the differences of temperature and concentration are of considerable interest in practice. Further, heat and mass transfer in the presence of magnetic field, which is the subject matter of magnetohydrodynamics (MHD), has different applications in natural phenomena and in many engineering problems. In recent times natural convective heat and mass transfer flows through a porous medium under the influence of a magnetic field have been paid attention of a number of a number of researchers because of their possible applications in many branches of science, engineering and geophysical process. Again suction is an important means to control the boundary layer development as well as to prevent the separation of flow. Thus, the effect of suction on MHD boundary layer is of great interest in astrophysics. Considering these numerous applications, MHD free convective heat and mass transfer flow in a porous medium have been studied by among others Raptis and Kafoussias (1982), Alam (1995) etc. Mohammed et al. (2005) investigated the effect of similarity solution for MHD flow through vertical porous plate with suction. Gundagani et al. (2013) presented a numerical solution to the problem of unsteady MHD free convective flow past a vertical porous plate with variable suction. Sattar et al. (2006) numerically studied a steady two-dimensional MHD free convective heat and mess transfer flow past an inclined semi-infinite surface with heat generation. Alam et al. (2013) studied heat and mass transfer in MHD free convection flow over an inclined plate with hall current. Alam et al. (2014) studied MHD boundary free conviction heat and mass transfer flow over an inclined porous plate with variable suction and Soret effect in presence of hall current. Rani et al. (2015) investigated the heat and mass transfer effect on MHD free convection flow over an inclined plate embedded in a porous medium. Recently, Opiyo et al. (2017) considered the effects of MHD on two dimensional steady free convection boundary layer heat and mass transfer flow of viscous, incompressible, electrically conducting fluid on an inclined plate with a varying angle of inclination.

In most of the above studies it was generally being considered that the magnetic Reynolds number is very small, so that the effect of induced magnetic field was neglected. But such effect must be considered for those cases in which values of magnetic Reynolds number is very high. However, Pantokratoras (2007) showed that a moving electrically conducting fluid induced a new magnetic field, which interacts with the applied external magnetic fields and the relative importance of this induced magnetic field depends on the relative value of the magnetic Reynolds number.

A numerical study of the natural convection heat and mass transfer about a vertical surface embedded in a saturated porous medium under the influence of induced magnetic field has carried out by Postelnicu (2004), taking into account the diffusion thermo and thermal diffusion effects. Using the shooting iteration numerical technique Alam et al. (2008) investigated the steady MHD heat and mass transfer by mixed convection flow from a moving vertical porous plate with induced magnetic, thermal diffusion, constant heat and mass fluxes. Recently, Bég et al. (2009) obtained closed-form local non-similarity numerical solutions for the velocity, temperature and induced magnetic field distributions in forced convection hydromagnetic boundary layers, over an extensive range of magnetic Prandtl numbers and Hartmann numbers. Applying the method of series solution, Ahmed and Chamkha (2010) investigated the effects of radiation and chemical reaction on steady mixed convective heat and mass transfer flow of an optically thin gray gas over an infinite vertical porous plate with constant suction taking into account the induced magnetic field, and viscous dissipation of energy. Ali et al. (2011) investigated the steady MHD mixed convection stagnation point flow over a vertical flat plate. Following the study to those of Pantokratoras (2007) and Postelnicu (2004), Hossain and Khatun (2012) investigated the Dufour effect on combined heat and mass transfer of a steady laminar mixed free-forced convective flow of viscous incompressible electrically conducting fluid above a semi-infinite vertical porous surface under the influence of an induced magnetic field. They have used the perturbation technique to solve the problem. Later, Hossain et al. (2013) studied the steady MHD free convection heat and mass transfer flow about a vertical porous surface with thermal diffusion in the presence of induced magnetic field. Khan et al. (2014) investigate the effects of heat generation, radiation and chemical reaction on unsteady mixed convection flow from a moving vertical porous plate with induced magnetic field, time dependent suction velocity at the plate, thermal diffusion, constant heat and mass fluxes. Asaduzzaman et al.

(2016) considered the transient heat transfer flow along a vertical plate with induced magnetic field.

The dual effects of the transverse magnetic field with the induced magnetic field on a steady mixed convective heat and mass transfer flow of an incompressible viscous electrically conducting fluid past an infinite vertical isothermal porous plate was studied by Ahmed and Zueco (2010). They have used the network simulation technique to solve the non-linear coupled equations taking into account the effects of viscous and magnetic dissipations of energy in presence of chemical reaction and heat generation/absorption. Using the Galerkin finite element method, an analysis was performed by Reddy and Rao (2011) to study the effect of thermal diffusion on an unsteady MHD free convective mass transfer flow of incompressible electrically conducting fluid past an infinite vertical porous plate with Ohmic dissipation. It was considered that the plate temperature oscillates with the same frequency as that of variable suction velocity and influence of uniform magnetic field is applied normal to the flow. Ahmed et al. (2012) investigated the effect of the transverse magnetic field on a transient free and forced convective flow over an infinite vertical plate impulsively held fixed in free stream taking into account the induced magnetic field. Wahiduzzaman et al. (2015) presented a numerical solution to investigate the influence of the hall current and constant heat flux on the MHD natural convection boundary layer viscous incompressible fluid flowing in the manifestation of transverse magnetic field near an inclined vertical permeable flat plate. In their analysis they assumed that the induced magnetic field is negligible compared with the imposed magnetic field.

On the other hand, an inclination of applied magnetic field is essential to explain the competency of magnetohydrodynamic plasma devices, accelerators, energy systems and also for the study of more pragmatic geophysical and biological flows. An inclined magnetic field is just a magnetic field which is applied with a nonzero inclination. In other words, the inclined magnetic field is the generalization of a magnetic field. The details of the influence of inclined magnetic fields on the flow of Newtonian and non-Newtonian fluids through different geometries have been presented by various researchers (Elshehawey et al.(2003), Nadeem and Akram (2010b) etc). Choudhary and Sharma

(2006) studied the laminar mixed convection flow of an incompressible electrically conducting vicious fluid over a continuously moving porous vertical plate with inclined magnetic field. Ahmed and Alam (2014)analyzed unsteady mixed convective ionized fluid flow through a vertical plate with joule heating, viscous dissipation, thermal diffusion, diffusion thermo, internal heat generation with chemical reaction, thermal radiation effects with an inclined uniform magnetic field in a rotating system. An investigation was performed by Reddy et al. (2015) in order to analyze the effects of aligned magnetic field on hydrodynamic free convection and mass transfer flow of viscous-elastic fluid through porous medium bounded by an oscillating porous surface in the slip flow regime with constant suction and temperature dependent heat source. The effect of thermal diffusion on the combined MHD heat transfer in an unsteady flow past a continuously moving semi-infinite vertical porous plate under the action of strong applied magnetic field has been investigated numerically by Islam et al. (2016)taking into account the induced magnetic field. Lastly, using Laplace Transform technique, Singh et al. (2016) have investigated the effect of inclined magnetic field on unsteady flow past a moving vertical plate with variable temperature.

Based on above stated studies, the present study deals with the study of steady two-dimensional MHD natural convection heat and mass transfer flow past an infinite vertical porous plate, taking into account the effects of thermal diffusion and large suction with inclined magnetic field.

The present thesis is arranged in the following pattern: Considering various aspects of an MHD heat and mass transfer flow, available information regarding MHD heat and mass transfer flows along with various effects are summarized and discussed in CHAPTER I from both analytical and numerical point of view. In CHAPTER II, the basic governing equations related to the problem considered thereafter are shown in the modified form and detailed calculation techniques for the problem are discussed. In CHAPTER III, an analytical solution of the problem based on perturbation technique has been discussed. In CHAPTER IV, the perturbation solutions and results discussions are presented. In CHAPTER V, the conclusions gained form this work are discussed.

CHAPTER I

Available Information on MHD Heat and Mass Transfer Flows

1.1 Magnetohydrodynamics (MHD)

Magentohydrodynamics (MHD) is the branch of magneto fluid dynamics, which deals with the flow of electrically conducting fluid in electric and magnetic field. The MHD phenomena are a consequence of the mutual interaction of the fluid flow and the magnetic field. Probably, the largest advancement towards an understanding of such phenomena comes from the field of astrophysics. It has long been suspected that most of the matter in the universe is in the form of plasma or highly ionized gaseous state and much of the basic knowledge in the area of electromagnetic fluid dynamics evolved from these studies.

The field of MHD consists of the study of a continuous, electrically conducting fluid under the influence electromagnetic fields. Originally, MHD included only the study of partially ionized gases as well as the other names have been suggested, such as magneto fluid mechanics or magneto aerodynamics, but the original nomenclature has persisted. The essential requirement for problem to be analyzed under the law of MHD is that the continuum approach be applicable. There are many natural phenomena and engineering problems susceptible to MHD analysis. It is the useful in astrophysics because much of the universe is filled with widely spaced charged particles and permeated by magnetic fields and so the continuum assumption becomes applicable. Engineers employ MHD principles in the design of heat exchangers, pumps and flow meters; in solving space vehicle propulsion, control and reentry problem; in designing communications and radar system; in creating novel power generating systems, and in developing confinement schemes for controlled fusion. The MHD in the generation of electrical power with the flow of electrically conducting fluid through as right-hand transverse magnetic field is one of the most important applications. Recently, this experiment with large magnetic fields is used for the generation of MHD power on a smaller scale is of interest of space applications.

Generally we know that, to convert the heat energy in to the electricity, several intermediate transformations are necessary. Each of these steps means a loss of energy. This naturally limits the overall efficiency, reliability and compactness of the conversion process. Method for the direct conversion to energy is now increasingly receiving attention. Of these, the fuel converts the chemical of fuel directly into electrical energy; fusion energy utilizes the energy released when two hydrogen molecules fuse into a heavier one, and thermoelectrically power generation uses a thermocouple. MHD power generation is another new process that has received worldwide attention. The principal MHD effects were first demonstrated in the experiments of Faraday (1832) find out experiments with flow of mercury in glass tubes placed between poles of a magnet and discovered that a voltage was induced across the tube by the motion of the mercury across the magnetic field, perpendicular to the direction of flow and to the magnetic field. Faraday observed that the current generated by this induced voltage interacted with the magnetic field to slow down the motion of the fluid, and he was aware of the fact that the current produced its own magnetic fluid that obeyed Ampere right-hand rule and thus, in turn distorted the field of magnet. Michael Faraday in 1832 discovered that when an electric field was applied to a conducting fluid perpendicularly to a magnetic field, it pumped the fluid in a direction perpendicular to both fields. Faraday also suggested that electrical power could be generated in a load circuit by the interaction of a flowing conduction fluid and a magnetic field.

The first astronomical application of the MHD theory occurred in 1899, when Bigalow suggested that the sun as a gigantic magnetic system. It remained, however, for Alfven (1942) to make a most significant contribution by discovering MHD waves in the sun. These waves are produced by disturbances which propagate simultaneously in the conducting fluid and the magnetic field. The analogy that explains the generation of an Alfven wave is that of a harp string plucked while submerged in a fluid. The string provides the elastic force and the fluid provides the inertia force, and they combine to propagate a perturbing wave through the fluid and the string.

It is well known that, a conductor when crossing magnetic field lines gives rise to an induced electric field, which drives an electric current in the conducting fluid. The resulting Lorentz force accelerates the fluid across the magnetic field, which in turn creates another induced electric field and currents which modify the initial magnetic field. Thus, the bulk motion of a conducting fluid and a magnetic field influence each other and must be determined self-consistently. In summary, MHD phenomena result from the mutual effect of magnetic field and conducting fluid flowing across it. Thus, an electromagnetic force is produced in a fluid flowing across a transverse magnetic field, and the resulting current and magnetic field combine to produce a force that resists the fluid's motion. The current also generates its own magnetic field which distorts the original magnetic field. An opposing or pumping force on the fluid can be produced by applying an electric field perpendicularly to the magnetic field. Disturbance in either the magnetic field or the fluid can propagate in both to produce MHD waves, as well as upstream and downstream-wake phenomena. The science of MHD is the detailed study of these phenomena, which occur in nature and are produced in engineering devices. Therefore, one of the major results of magnetohydrodynamics is the ability of conducting fluids to amplify magnetic fields and the amplification of these magnetic fields being a universal necessity.

1.2 Electromagnetic Equations

MHD equations are the ordinary electromagnetic and hydrodynamic equations which have been modified to take account of the interaction between the motion of the fluid and electromagnetic field. The basic laws of electromagnetic theory are all contained in special theory of relativity. But it is always assumed that all velocities are small in comparison to the speed of light. Before writing down the MHD equations we will first of all notice the ordinary electromagnetic equations (Cramer and Pai (1973)). The mathematical formulation of the electromagnetic theory is known as Maxwell's equations which explore the relation of basic field quantities and their production. The Maxwell's electromagnetic equations are given by

Charge continuity $$\nabla \cdot \mathbf{D} = \rho_c \tag{1.1}$$

Current continuity $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho_c}{\partial t}$ (1.2)

Magnetic field continuity $\nabla \cdot \mathbf{B} = 0$ (1.3)

Ampere's law $\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{J\mathbf{D}}{\partial t}$ (1.4)

Faraday's law $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (1.5)

Constitutive equations for $\mathbf{D}$ and $\mathbf{B}$ are $\mathbf{D} = \varepsilon' \mathbf{E}$ (1.6)

$\mathbf{B} = \mu_c \mathbf{H}$ (1.7)

Lorentz force on a change $\mathbf{F}_p = q'\left(\mathbf{E} + \mathbf{q}_p \times \mathbf{B}\right)$ (1.8)

Total current density flow $\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{q} \times \mathbf{B}\right) + \rho_c \mathbf{q}$ (1.9)

In equations (1.1)–(1.9), $\mathbf{D}$ is the displacement current, ρ_c is the charge density, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the induced magnetic field. $\mathbf{E}$ is the electric field, ε' is the electrical permeability of the medium μ_c is the magnetic permeability of the medium, $\mathbf{q}_p$ is the velocity of the charge, σ is the electrical conductivity, $\mathbf{q}$ is the velocity of the fluid and $\rho_c \mathbf{q}$ is the convection current due to charges moving with the fluid.

1.3 Fundamental Equations of fluid Dynamics of Viscous Fluids

In the study of fluid flow one determines the velocity distribution as well as the states of the fluid over the whole space for all time, there are six unknowns namely, the three components (u, v, w) of velocity $\mathbf{q}$, the temperature T, the pressure p and the density ρ of the fluid, which are function of special co-ordinates and time. In order to determine these unknown we have the flowing equation:

(a) Equation of state, which connects the temperature, the pressure and density of the fluid is given by $p = \rho RT$ (1.10)

For an incompressible fluid the above equation is simply $\rho = \text{const.}$ (1.11)

(b) Equation of continuity, which gives relation of conservation of mass of the fluid. The equation of continuity for a viscous incompressible fluid is

$$\nabla \cdot \mathbf{q} = 0 \tag{1.12}$$

(c) Equation of motion, also known as the Navier-Stokes' equations, which give the relations of the conservation of momentum of the fluid. For viscous incompressible fluid the equation of motion is

$$\rho \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{q} \tag{1.13}$$

Where **F** is the body force per unit volume and the last term on the right hand side represents the force per unit volume due to viscous stresses and p is the pressure. The operator,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

is known as the material derivative or total derivative with respect to time which gives the variation of a certain quantity of the fluid particle with respect to time. Also ∇^2 represents the Laplacian operator.

(d) The equation of energy, which gives the relation of conservation of energy of the fluid. For an incompressible fluid with constant viscosity and heat conductivity, the energy equation is

$$pC_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\nabla^2 T + \phi \tag{1.14}$$

C_p is the specific heat at constant pressure, $\frac{\partial Q}{\partial t}$ is the rate of heat produced per unit volume by external agencies, k is the thermal conductivity of the fluid, ϕ is the viscous dissipation function for an incompressible fluid $\phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(Y_{xy}^2 + Y_{yz}^2 + Y_{zx}^2\right)\right]$

Where $Y_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $Y_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ and $Y_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$.

(e) The concentration equation for viscous incompressible fluid is

$$\frac{DC}{Dt} = D_m \nabla^2 C \tag{1.15}$$

where C is the concentration and D_m is the chemical molecular diffusivity.

1.4 MHD Approximations

The electromagnetic equation as given in (1.1) – (1.9) are not usually applied in their present form and requires interpretation and several assumptions to provide the set to be used in MHD. In MHD we consider a fluid that is grossly neutral. The charge density ρ_c in Maxwell's equations must then be interpreted as an excess charge density which is in general not large. If we disregard the excess charge density, we must disregard the displacement current. In most problems due to convection of the excess charge are small (Cramer and Pai, et al (1973)).

The electromagnetic equations to be used are then as follows:

$$\nabla \cdot \mathbf{D} = \mathbf{0} \tag{1.16}$$

$$\nabla \cdot \mathbf{J} = \mathbf{0} \tag{1.17}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0} \tag{1.18}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.19}$$

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{1.20}$$

$$\mathbf{D} = \varepsilon' \mathbf{E} \tag{1.21}$$

$$\mathbf{B} = \mu_c \mathbf{H} \tag{1.22}$$

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{q} \times \mathbf{B} \right) \tag{1.23}$$

1.5 MHD Equations

We will now modify the equations of fluid dynamics suitably to take account of the electromagnetic phenomena.

(a) The MHD equation of continuity for viscous incompressible electrically conducting fluid remains the same

$$\nabla \cdot \mathbf{q} = 0 \tag{1.24}$$

(b) The MHD momentum equation for a viscous incompressible and electrically conducting fluid is

$$\rho \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{q} + \mathbf{J} \times \mathbf{B} \tag{1.25}$$

Where $\mathbf{F}$ is the body force per unit volume corresponding to the usual viscous fluid dynamic equations and the new term $\mathbf{J} \times \mathbf{B}$ is the force on the fluid per unit volume produced by the interaction of the current and magnetic field (called a Lorentz force).

(c) The MHD energy equation for a viscous incompressible electrically conducting fluid is

$$\rho C_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\nabla^2 T + \phi + \frac{J^2}{\sigma} \tag{1.26}$$

The new term is the Joule heating term and is due to the resistance of the fluid to the flow of current.

(d) The MHD equation of concentration for viscous incompressible electrically conducting fluid remains the same as

$$\frac{DC}{Dt} = D_m \nabla^2 C \tag{1.27}$$

1.6 Some Useful Dimensionless Numbers/Parameters

1.6.1 Reynolds number (R_e)

The Reynolds number R_e is the most important parameter of the fluid dynamics of a viscous fluid, which is defined by the following ratio

$$R_e = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\text{Mass}\times\text{Acceleration}}{\text{Shear stress}\times\text{Cross sectional area}} = \frac{\rho L^3 \times \frac{U}{T}}{\mu \times \frac{U}{T} \times L^2} = \frac{\rho LU}{\mu} = \frac{LU}{\upsilon}$$

where, L and U denotes the Characteristic length and velocity respectively and $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity (μ is the viscosity and ρ is the density).

For if R_e is small, the viscous force will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand if R_e is large the inertia force will be predominant and in such case the effect of viscosity to be confined in a thin layer, near to the solid wall or other restricted region, which is known as boundary layer. However if R_e is very large, the flow ceases to be laminar and becomes turbulent. The Reynolds number at which translation from laminar to turbulent occurs is known as critical Reynolds number.

Reynolds in 1883 found that for flow in a circular pipe becomes turbulent when R_e exceeds the critical value 2300, i.e. $R_e = \left[\frac{\bar{U}\, d}{\upsilon}\right]_{crit} = 2300$, where $\bar{U}$ is the mean velocity and '*d*' is the diameter of the pipe. When the viscous force is pre-dominating force, Reynolds number must be similar for dynamic similarity of two flows.

1.6.2 Prandtl number (P_r)

The Prandtl number is a dimensionless number, named after its inventor, a German engineer Ludwig Prandtl, who also identified the boundary layer. The Prandtl number (P_r) is the ratio of momentum diffusivity to thermal diffusivity. The momentum diffusivity, or as it is normally called, kinematic viscosity, tells us the material's resistance to shear-flows (different layers of the flow travel with different velocities e.g. different speeds of adjacent walls) in relation to density. Thus the Prandtl number is defined as

$$P_r = \frac{\text{Viscous diffusion rate}}{\text{Thermal diffusion rate}} = \frac{\upsilon}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho c_p}} = \frac{\mu c_p}{k}$$

Where c_p is the specific heat at constant pressure μ is dynamic viscosity of the fluid and k is the thermal conductivity. The value of $\frac{k}{\rho c_p}$ is the thermal diffusivity due to the heat condition. The smaller value of $\frac{k}{\rho c_p}$ is, the narrower is the region which affected by the heat condition and it is known as the thermal boundary layer, the value of $\upsilon = \frac{\mu}{\rho}$ show the effect of viscosity of the fluid. Thus, the Prandtl number shows that the relative importance of heat conduction and viscosity of a fluid. Small values of the Prandtl number (P_r $\ll 1$) means the thermal diffusivity dominates. Whereas, with large values of Prandtl number ($P_r \gg 1$), the momentum diffusivity dominates the behavior. Evidently P_r varies from fluid to fluid. For example, the typical value for liquid mercury, which is about 0.025, indicates that the heat conduction is more significant compared to convection, so thermal diffusivity is dominant. When Pr is small, it means that the heat diffuses quickly compared to the velocity. For air $P_r = 0.72$ (approx), for water at 15.5^0c, $P_r = 7.00$ (approx), for mercury $P_r = 0.044$, but for high viscous fluid it may be very large, e.g. for glycerin $P_r = 7250$.

1.6.3 Magnetic Force number (M)

The magnetic force number is the ratio of the magnetic force to the inertia force and is defined by

$$M = \frac{\text{Magnetic force}}{\text{Inertia force}} = \frac{\mu_3^e H_0^2 \sigma' L}{\rho U}$$

1.6.4 Schmidt number (S_c)

The Schmidt number is a dimensionless number which is the ratio of momentum diffusivity or kinematics viscosity to the mass diffusivity. In other sense Schmidt number is the ratio of the viscous diffusivity to the chemical molecular diffusivity. It was named after the German engineer Ernst Heinrich Wilhelm Schmidt (1892-1975) and is defined by

$$S_c = \frac{\text{kinematics viscosity}}{\text{mass diffusivity}} = \frac{\text{Viscous diffusivity}}{\text{Chamical molecular diffusivity}} = \frac{\upsilon}{D_m} = \frac{\mu}{\rho D_m}$$

1.6.5 Grashof number (G_r)

The Grashof number is a dimensionless number which approximates the ratio of the buoyancy force to the viscous force acting on a fluid. The Grashof number frequently arises in the study of physical parameters or in the situations involving natural convection. It is named after the German engineer Franz Grashof. The Grashof number is defined as $G_r = \frac{\text{Buoyancy force}}{\text{Viscous force}} = \frac{g\beta\Delta T L^3}{\upsilon^2}$, where g is the acceleration due to earth's gravity, β is the thermal expansion coefficient, L is the distance between region of high and low temperature (Characteristic length), ΔT is temperature difference and υ is the kinematic viscosity. Thus, it measures of the relative importance of the buoyancy and viscous forces. The larger it is, stronger is the convective current. At higher Grashof numbers, the boundary layer is turbulent and at lower Grashof numbers, the boundary layer is laminar.

1.6.6 Modified Grashof number $\left(G_m\right)$

The modified Grashof number usually occurring in free convection problem, when the effect of mass transfer is also considered. The modified Grashof number is defined by $G_m = \dfrac{g\beta^* \Delta C L^3}{\upsilon^2}$, where β^* is the volumetric co-efficient of thermal expansion or the concentration expansion coefficient and ΔC be the concentration difference.

1.6.7 Soret number $\left(S_0\right)$

The Soret number is defined by $S_0 = \dfrac{D_T\left(T_w - T_\infty\right)}{\upsilon\left(C_w - C_\infty\right)}$

1.6.8 Magnetic diffusivity parameter $\left(P_m\right)$

The magnetic diffusivity is defined by $P_m = \mu_e \sigma' \upsilon$

1.6.9 Eckert number $\left(E_c\right)$

The Eckert number is defined by $E_c = \dfrac{U^2}{c_p\left(T_w - T_\infty\right)}$

1.7 Suction and Injection

For ordinary boundary layer flows of adverse pressure gradients, the boundary layer flow will eventually separate from the surface. Separation of the flow causes many undesirable features over the whole field, consequently, it is often necessary to prevent separation of the boundary layer to reduce the drag forces and attain high lift values. For instance, if separation occurs on the surface of an airfoil, the lift of the airfoil will decrease and the drag will enormously increase. In some problems it is very essential to maintain laminar flow through preventing separation. To prevent the separation of boundary layer flows, various ways have been proposed of which suction/injection are very important ones. Besides, the stabilizing effect of the boundary layer development has been well known for several years and till to date suction/injection are still the most of efficient, simple and common method of boundary layer control. Hence, the effect of suction on hydromagnetic boundary layer is of great interest in astrophysics.

Many authors have made mathematical studies on these problems, specially in the case of steady flow. Among them the name of Cobble (1977)may be cited who obtained the conditions under which similarity solutions exist for hydromagnetic boundary layer flow past a semi-infinite flat plate with or without suction. Following this, Soundalgekar and Ramanamurthy (1980) analyzed the thermal boundary layer. Then Singh (1980) studied this problem for large values of suction velocity employing asymptotic analysis in the spirit of Nanbu (1971).Singh and Dikshit (1988) have again adopted the asymptotic method to study the hydromagnetic effect on the boundary-layer development over a continuously moving plate. In a similar way Bestman (1990) studied the boundary layer flow past a semi-infinite heated porous plate for two component plasma.

On the other hand, one of the important problems facing the engineers engaged in high speed flow is the cooling of the surface to avoid the structural failures as a result of frictional heating and other factors. In these respect the possibility of using injection at the surface is a measure to cool the body in the high temperature fluid. Injection of secondary fluid through porous walls is of practical importance in film cooling of turbine blades combustion chambers. In such applications injection usually occurs normal to the surface and the injected fluid may be similar to or different from the primary fluid. In some recent applications, however, it has been recognized that the cooling efficiency can be enhanced by vectored injection at an angle other than 90^0 to the surface. Kay (1953) studied the boundary layer growth on an infinite flat plate with uniform suction or injection. Exact solutions of the Navier-stokes' equations of motion were derived for the case uniform suction and injecting which was taken to be steady or proportional to $t^{-1/2}$ and the plate is perfectly insulated, that is, there is no heat transfer between the fluid and the plate. Numerical calculations are also made for the case of impulsive motion of the plate. Raptis and Shing (1983) studied the free convection effects on the flow field of an incompressible, viscous dissipative fluid, past an infinite vertical porous plate which is accelerated in its own plane. The fluid is subjected to a normal velocity of suction/injection which is inversely proportional to the square root of time (i.e. $t^{-1/2}$) and the plate is perfectly insulated, i.e., there is no heat transfer between the fluid and the plate. The qualitative natures of the flow for the case of both suction and injection are sometimes the same which are obtained from the results of the corresponding studies on steady boundary layer.

1.8 MHD Boundary Layer and Related Transfer Phenomena

Boundary layer phenomena occurs when the viscous effect may be considered to be confined in a very thin layer near to the boundaries and the non-dimensional diffusion parameter such as the Reynolds number, the Pe'clet number and the Magnetic Reynolds number are very large. The boundary layers are then the velocity and thermal (or magnetic) boundary layers and each of its thickness is inversely proportional to the square root of the associated diffusion number. Prandtl observed form experimental flows that, in classical fluid dynamics boundary layer theory, for large Reynolds number, the viscosity and the thermal conductivity appreciably influences the flow only near a wall. When distance measurements in the flow direction are compared with a characteristic dimension in that direction, transverse measurement compared with the boundary layer thickness and velocities compared with the free stream velocity, the Navier-Stokes and energy equations can be considerably simplified neglecting small quantities. The flow directional component equations only remain and pressure is then only a function of the flow direction and can be determined from the non-viscous flow solution. Also the number of viscous term is reduced to the dominant term and the heat condition flow direction is negligible. Therefore, two types of MHD boundary layer flows are found depending on the limiting cases of a very large and a negligible small Magnetic Reynolds number. When the magnetic Reynolds number is large, the magnetic boundary layer thickness is small and is of nearly the same size of the viscous and thermal boundary layers and then the equations of the MHD boundary layer must be solved simultaneously. On the other hand, when the magnetic Reynolds number is very small and the magnetic field is oriented in an arbitrary direction relative to a confining surface; the flow direction component of the magnetic interaction and the corresponding joule heating is only a function of the transverse magnetic field component and the local velocity in the flow direction. Changes in the transverse magnetic boundary layer are negligible. The thickness of the magnetic boundary layer is very large and the induced magnetic field is negligible. In this case the magnetic field moves with the flow and is called frozen mass.

1.9 MHD and Heat Transfer

With the advent hypersonic flight, the field of MHD, as defined above, which has been associated largely with liquid metal pumping, has attracted the interest of aerodynamics. It is possible to alter the flow and the heat transfer around high-velocity provided that the air is sufficiently ionized. Furthermore, the invention of high temperature facilities such as the shock tube and plasma jet has provided laboratory sources of flowing ionized gas, which provide an incentive for the study of plasma accelerators and generators.

As a result of this, many of the classical problems of fluid mechanics have been reinvestigated. Some of these analyses arose out of the natural tendency of scientists to investigate a new subject. In this case it was the academic problem of solving the equations of fluid mechanics with a new body force and another source of dissipation in the energy equation were presented. Sometimes there were no practical applications for these results. For Example, natural convection MHD flows have been of interest to the engineering community only since the introduction of liquid metal heat exchangers, whereas the thermal instability investigations are directly applicable to the problems in geophysics and astrophysics.

1.10 Free and Force Convection

In the studies related to heat transfer, considerable effort has been directed towards the convective mode, in which the relative motion of the fluid provides an additional mechanism for the transfer of energy and material, the later being a more important consideration in cases where mass transfer, due to a concentration difference, occurs. Convection is inevitably coupled with the conductive mechanisms, since, although the fluid motion modifies the transport process, the eventual transfer of energy from one fluid element to another in its neighborhood is thorough conduction. Also, at the surface the process is predominantly that of conduction because the relative fluid motion is brought to zero at the surface. A study of the convective heat transfer therefore involves the mechanisms of conduction and sometimes those of radioactive processes as well, couples with that fluid flow. These make the study of this mode of heat or mass transfer very complex, although its importance in technology and in nature can hardly be exaggerated. The heat transfer in convective mode is divided into two basic processes.

Free convection is sometimes known as natural convection. It is the natural flow of air or fluid over a surface without any external driving force. Free convection is a mechanism or type of heat transfer, in which the motion is generated without any action of external source (like a pump, fan and suction device etc.) and flow arises naturally simply owing to the effect of a density difference, resulting from a temperature or concentration difference in a body force field, such as the gravitational field, the process is referred to the natural convection. In the natural convection, the density difference gives rise to buoyancy effects, owing to which the flow is generated. A heated body cooling in ambient air generates such a flow in the region surrounding it. Another classical natural convection problem is the thermal instability that occurs in a liquid heated from below. In such a case, fluid neighboring a heat source receives heat, becomes less dense and rises then the neighboring cooler fluid moves to replace it. This cooler fluid is then heated and the process continues, forming a current convection; this process transfer heat energy from the bottom of the convection cell to the top. This subject is of natural interest to geophysicists and astrophysicists, although some applications might arise in boiling heat transfer. The driving force for natural convection is buoyancy, a result of differences in fluid density. It does not appear in inertial or free-fall environment. It has attracted a great deal of attention from researcher because of its presence both in nature and industrial applications. Convection is also seen in the rising plume of hot air from fire, oceanic currents and sea formation. A very common industrial application of natural convection is free air cooling without the aid of fans. The flow may also arise owing to concentration differences such as those caused by salinity differences in the sea and by composition differences in chemical processing unit, and these cause a natural convection mass transfer.

Forced convection is just what it says. If the motion of the field is caused by an external agent such as the externally imposed flow of a fluid stream over a heated object, the process is termed as force convection. It is a mechanism, or type of transport in which fluid motion is generated by an external source (like a pump, fan, suction device, etc.). It is caused due to the forced flow of a fluid or air over a surface. In the force convection, the fluid flow may be the result of, for instance, a fan, a blower, the wind or the motion of the heated object itself. Such problems are very frequently encountered in technology where the heat transfers to or from a body is often due to an imposed flow of a fluid at a different temperature from that of a body. It should be considered as one of the main methods of useful heat transfer as significant amounts of heat energy can be transported very efficiently. This mechanism is found very commonly in everyday life, including central heating, air conditioning, steam turbines and in many other machines. Forced confection is often encountered by engineers designing or analyzing heat exchangers, pipe flow, and flow over a plate at a different temperature than the stream (the case of a shuttle wing during re-entry, for example). In any forced convection situation, some amount of natural convection is always present whenever there is g-forces present (i.e., unless the system is in free fall).Practically some time both processes, natural and forced convection are important and heat transfer is occurred by mixed convection, in which neither mode is truly predominant. The main difference between the two really lies in the word external. A heated body lying in still air loses energy by natural convection. But it also generates a buoyant flow above it and body placed in that flow is subjected to an external flow and it becomes necessary to determine the natural, as well as the forced convection effects in the regime in which the heat transfer mechanisms lie. When MHD become a popular subject, it was normal that these flows would be investigated with the additional ponder motive body force as well as the buoyancy force. At a first glance there seems to be no practical applications for these MHD solutions, for most heat exchangers utilize liquids, whose conductively is so small that prohibitively large magnetic fields are necessary to influence the flow. But some nuclear plants employ heat exchangers with liquid metal coolants, so the application of moderate magnetic fields to change the convection pattern appears feasible. Another classical natural convection problem is the thermal instability that occurs in a liquid heated from below. This subject is of natural to geophysicists and astrophysicists, although some applications might arise in boiling heat transfer.

The basic concepts involved in employing the boundary layer approximation to natural convection flows are very similar to tide in forced flows. The main difference lies in the fact is that the pressure in the region beyond the boundary layer is hydrostatic instead of being imposed by an external flow, and that the velocity outside the layer is zero, however, the basic treatment and analysis remain the same.

The book by Schlichting (1968) is an excellent collection of the boundary layer analysis. There are several method for the solution of the boundary layer equations, namely, the similarity variable method, the perturbation method, analytical method, numerical method etc. and their details are obtained in the books by Rosenberg (1969), Patanker and Spalding (1970) and Spalding (1977).

1.11 Heat and Mass Transfer

The basic heat and mass transfer problem is governed by the combined buoyancy effects arising from the simultaneous diffusion of thermal energy and chemical species. Therefore the equations of continuity, momentum, energy, mass diffusion are coupled through the buoyancy terms alone, if there are other effects, such as the Soret and Duffor effects, they are neglected. This would again valid for low species concentration levels. These additional effects have also been considered in several investigations, for example, the work of the Groots and Mozur (1962), Caldwell (1974) etc.

Somers (1956) considered combined buoyancy mechanisms for flow adjacent to a wet isothermal vertical surface in an unsaturated environment. Uniform temperature and uniform species concentration at the surface were assumed and an integral analysis was carried out to obtain the result which is expected to be valid for the values of Prandtl number ($\Pr$) and Schmidt number (Sc) around 1.0 with one buoyancy effect being small compared with the other. Gebhart and Pera (1971) studied laminar vertical natural convection flows resulting from the combined buoyancy mechanisms in terms of similarity solutions. Georgantopolous et al. (1979) have studied the effects of mass transfer on free convection problem in the stokes' problem for an infinite vertical limiting surface. Raptis and Kafoussias (1982) presented the analysis of free convection and mass transfer steady hydromagnetic flow of an electrically conducting viscous incompressible fluid, through a porous medium, occupying a semi infinite region of the space bounded by an infinite vertical porous plate under the action of transverse magnetic field. Agrawal et al. (1983) have investigated the effect of Hall current on the combined effect of thermal and mass diffusion of an electrically conducting liquid past an infinite vertical porous plate, when the free stream oscillates about constant non zero mean. The velocity and temperature distributions are shown on graphs for different values of obtained parameters.

1.12 Thermal Diffusion

In the above mentioned studied heat and mass transfer occur simultaneously in a moving fluid, where the relations between the fluxes and driving potentials are of more complicated nature. In general the thermal diffusion effects is of a small order of magnitude, described by Fourier or Flick's law, is often neglected in heat and mass transfer processes. Mass fluxes can also be created by temperature gradients and this is Soret or thermal diffusion effect. There are however, exceptions. The thermal diffusion effect, for instance, has been utilized for isotope separation and in mixtures between gases with very light molecular weight (H_2, He) and of medium molecular weight (N_2, air). Kafoussias (1992) studied the MHD free convection and mass transfer flow, past an infinite vertical plate moving on its own plane, taken into account the thermal diffusion when (i) the boundary surface is impulsively started moving in its own plane (I.S.P) and (ii) it is uniformly accelerated (U.A.P). The problem was solved with the help of Laplace transformation method and analytical expressions were given for the velocity and skin friction for the above-mentioned two different cases. The effect of the velocity and skin friction of the various dimensionless parameters entering into the problem was discussed with the help of graph. For both the cases, it was seen from the figure that the effect of magnetic parameters M is to decreases the fluid (water) velocity inside the boundary layer. This influence of the magnetic field on the velocity field is more evident in the presence of the thermal diffusion. From the same figures it is also concluded that the fluid velocity rises due to greater thermal diffusion. Hence, the velocity field is considerably affected by the magnetic field and the thermal diffusion. Nanousis (1992) extended the work of Kafoussias (1992) to the case of rotating fluid taking into account the Soret effect. The plate is assumed to be moving on its own plane with arbitrary velocity $U_0 f\left(t'\right)$ where U_0 is a constant velocity and $f\left(t'\right)$ is a non-dimensional function of time t'. The solution of the problem is given for the velocity field and for the skin friction for two different cases mentioned above.

1.13 Important Physical Parameters

Generally a parameter is any characteristic that can help in defining or classifying a particular event, project, object, or situation, etc. It is an element of a system that is useful, or vital, when evaluating the identify of a system; or, when evaluating performance, status, condition, etc. of a system that controls what something is or how something should be done. In this clause following physical parameters are described regarding our study:

1. Skin-friction co-efficient
2. Nusselt number.
3. Sherwood number

1.13.1 Skin-friction Coefficient (C_f)

Friction is the force resisting the relative motion of solid surfaces, fluid layers, and material elements sliding against each other. There are several types of friction such as skin friction, fluid friction and internal friction etc. Skin friction arises from the interaction between the fluid and the skin of the body, and is directly related to the area of the surface of the body that is in contact with the fluid. The dimensionless sharing stress on the surface of a body, due to fluid motion is known as skin-friction coefficient and it is defined as $C_f = \frac{\tau_w}{\rho {U_w}^2/2}$. Here τ_w is the local shearing stress on the surface of the body and it is defined as $\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$

1.13.2 Nusselt Number (N_u)

Nusselt number is the ratio of heat flow rate by convection under unit temperature gradient to the heat flow by conduction process under unit temperature gradient. Named after Wilhelm Nusselt, it is a dimensionless number. Nusselt number is defined as $N_u = -\frac{q_w L}{k\Delta T}$, where $q_w = k\left(\frac{\partial T}{\partial y}\right)_{y=0}$ is the wall heat flux and L is the characteristic length.

1.13.3 Sherwood Number (S_h)

The Sherwood number is a dimensionless number for convective mass transfer in fluid flow region. It represents the ratio between mass transfer coefficient, that is, mass transfer by convection and mass transfer by diffusion. It is named in the honor of Thomas Kilgore Sherwood.

Sherwood number is defined as $S_h = \frac{kL}{D}$, where k is mass transfer coefficient, D is the mass diffusivity, and L is representative dimension (e.g. diameter of Tubes or pipes). For example, if mass transfer coefficient k = 5 *m/s* Characteristic length L = 15 *m* then we have the Sherwood number $S_h = 7.5$. More conveniently, the ratio of mass transfer in terms of Sherwood in terms of Sherwood number at the plate is also defined as $S_h = \frac{m_w L}{D\Delta C}$, where $m_w = D\left(\frac{\partial C}{\partial \mathrm{y}}\right)_{\mathrm{y}=0}$ is the quantity of mass transfer through the unit area of the surface.

CHAPTER II

Basic Equations and Transformations

2.1 Governing Equations of the Flow

We consider the flow model of steady free convection heat and mass transfer flow of a viscous, incompressible, electrically conduction fluid past a semi-infinite vertical electrically non-conducting moving porous plate with thermal diffusion.

Following the Cartesian coordinate system, the *x*-axis is assumed to be in the upward direction, which is taken along the vertical porous plate in the direction of the flow and *y*-axis is normal to it. The leading edge of the plate is taken as coincident with *z*-axis .A strong uniform transverse magnetic field **B** is chosen to be applied in a direction that makes an angle α with the normal to the considered plate (i.e. the *y*-axis). Therefore, **B** is taken in the form $\mathbf{B} = \left(0, B_0 \cos\alpha, B_0\sqrt{1-\cos^2\alpha}\right)$. Thus, if $\cos\alpha = 1$, i.e. $\alpha = 0^0$, the imposed magnetic field is parallel to *y*-axis and if $\cos\alpha = 0$, i.e. $\alpha = 90^0$, then the magnetic field is parallel to the plate. Since the magnetic Reynolds number of the flow is not taken to be of small magnitude, the induced magnetic field is not negligible, so that it is taken into account with the applied magnetic field.

The physical model of this study as well as the coordinate system is furnished in the following figure (Figure 2.1.1):

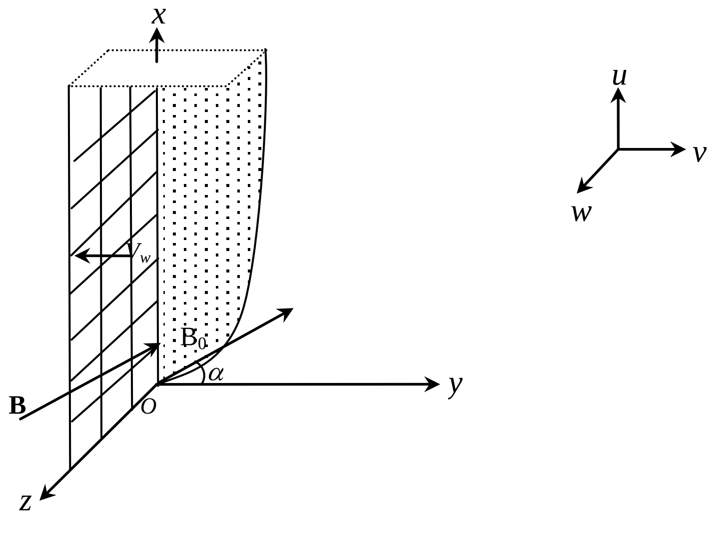

Figure 2.1.1: Physical configuration and the coordinates system.

Following the Cartesian coordinate system, the flow is assumed to be in the *x*-direction, which is taken along the vertical plate in the upward direction, where, *y*-direction is normal to the plate. The leading edge of the plate is taken as coincident with *z*-axis. A strong uniform transversely applied magnetic field of strength $B_0$is applied in the direction at an angle α with the *y*-axis. It is also assumed that all the physical properties of the field are considered to be constant except only the influence of variation of density due to the variation of temperature are coupled with the body force term, in accordance with the Boussinesq's approximation. Besides, no chemical reaction takes place between the foreign mass and the fluid considered. The induced magnetic field is of the form $\mathbf{H} = \left(H_x, H_y, 0\right)$. The equation of the conservation of electric charge is $\nabla \cdot \mathbf{J} = 0,$ where the current density $\mathbf{J} = \left(J_x, J_y, 0\right).$ Since the direction of propagation of electric charge is considered only along the *y*-axis and $\mathbf{J}$ does not have any variation along y axis, that is $\frac{\partial J_y}{\partial y} = 0 \Rightarrow J_y =$ constant and as the plate is electrically non-conducting, hence the constant is zero $J_y = 0$ at the plate as well as everywhere within the flow. Further the Joule heating effect is considered to be small enough. As the direction of propagator is along the y-axis and $\mathbf{H} = \left(H_x, H_y, 0\right)$ does not have any variation along y-axis, so that from diverge equation $\nabla \cdot \mathbf{H} = 0 \Rightarrow \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$ of Maxwell's equation for the magnetic field we have $\frac{\partial H_y}{\partial y} = 0$ i.e. $H_y =$ constant $= H_0$ (say) is the constant induced magnetic field. In accordance with the above stated assumptions, the generalized form of basic governing equations related to the problem with the incorporation of the Maxwell's equations and generalized Oham's law can be put in the following form:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} + g\beta\left(T - T_\infty\right) + g\beta^*\left(C - C_\infty\right) + \frac{B_0 \cos\alpha}{\rho}\frac{\partial H_x}{\partial y} \tag{2.2}$$

Magnetic induction equation:

$$u\frac{\partial H_x}{\partial x} + v\frac{\partial H_x}{\partial y} = H_x\frac{\partial u}{\partial x} + \frac{B_0 \cos\alpha}{\mu_e}\frac{\partial u}{\partial y} + \frac{1}{\sigma\mu_e}\frac{\partial^2 H_x}{\partial y^2} \tag{2.3}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\upsilon}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{\sigma\rho C_p}\left(\frac{\partial H_x}{\partial y}\right)^2 \tag{2.4}$$

Species (concentration) equation:

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}+D_T\frac{\partial^2 T}{\partial y^2} \tag{2.5}$$

The relevant boundary conditions on the vertical surface and in the uniform stream are defined as follows:

$$\left.\begin{array}{l} u=U_0,\, v=-V_0,\, T=T_w, C=C_w,\, H_x=H_w \text{ at } y=0 \\ u=0,\, v=0,\, T=T_\infty,\, C=C_\infty,\, H_x=0 \text{ when } y\to\infty \end{array}\right\} \tag{2.6}$$

where g is the acceleration due to gravity, β is the coefficient of thermal expansion, T denotes fluid temperature, C is concentration of species, T_∞ and C_∞ are the temperature and species concentration of the uniform flow, $\beta*$ is the concentration expansion coefficient, υ is the Newtonian kinematics viscosity of the fluid, μ_e is the magnetic permeability, H_0 is the constant magnetic field, H_x is induced magnetic field coefficient, ρ is the density of the fluid, σ is the electrical conductivity, C_p is the specific capacity of the fluid at constant pressure, C_w is concentration of species at the wall, k_T is the thermal diffusion ratio, U_0 is the uniform velocity of the plate which is taken as the free stream velocity, H_w is the induced magnetic field at the wall and V_0 is the non-zero suction velocity perpendicular to the wall (the negative sign indicates that the suction velocity is directed towards the plate surface), respectively. In order to simplify the above equations (2.1)–(2.5) with boundary conditions (2.6), we will now introduce the following transformations, viz:

$$\eta=y\sqrt{\frac{U_0}{2\upsilon x}},\; f(\eta)=\int_0^\eta\frac{u}{U_0}d\eta, \text{so that } f'(\eta)=\frac{u}{U_0},\, \theta=\frac{T-T_\infty}{T_w-T_\infty},\, \varphi=\frac{C-C_\infty}{\overline{x}\left(C_0-C_\infty\right)},$$

and $H_x=\sqrt{\dfrac{\rho}{\mu_e}}\sqrt{\dfrac{\upsilon U_0}{2x}}H(\eta)$ (2.7)

The equation of continuity i.e., equation (2.1) is identically satisfied by introducing the stream function ψ such that $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$.

Now from equation (2.7) we have $u=U_0f'(\eta)$.

Then by equation (2.1) we obtain $v=\sqrt{\dfrac{\upsilon U_0}{2x}}\{\eta f'(\eta)-f(\eta)\}$

2.2 Mathematical Calculations

Calculation for equation (2.2):

$$u\frac{\partial u}{\partial x}=-\frac{U_0^2\eta}{2x}f''(\eta)f'(\eta) \tag{2.8}$$

$$v\frac{\partial u}{\partial y}=\frac{U_0^2}{2x}\left[\eta f'(\eta)-f(\eta)\right]f''(\eta) \tag{2.9}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{U_0^2}{2\upsilon x}f'''(\eta)$$

Therefore, $\upsilon\dfrac{\partial^2 u}{\partial y^2}=\dfrac{U_0^2}{2x}f'''(\eta)$ (2.10)

$$H_x=\sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2x}}H(\eta)$$

$$\frac{\partial H_x}{\partial y}=\frac{U_0}{2x}\sqrt{\frac{\rho}{\mu_e}}H'(\eta) \tag{2.11}$$

Now using equations (2.8)–(2.11) in equation (2.2) we get

$$-\frac{U_0^2\eta}{2x}f''(\eta)f'(\eta)+\frac{U_0^2}{2x}\left\{\eta f'(\eta)-f(\eta)\right\}f''(\eta)=\frac{U_0^2}{2x}f'''(\eta)+g\beta\left(T-T_\infty\right)$$
$$+g\beta^*\left(C-C_\infty\right)+\frac{B_0}{\mu_e}\frac{\cos\alpha U_0}{2x}\sqrt{\frac{\mu_e}{\rho}}H'(\eta)$$

or $f'''(\eta)+f(\eta)f''(\eta)+\dfrac{g\beta\left(T_w-T_\infty\right)2x}{U_0^2}\dfrac{\left(T-T_\infty\right)}{\left(T_w-T_\infty\right)}+\dfrac{g\beta^*\left(C_0-C_\infty\right)2x^2}{U_0^2}$

$$\frac{\left(C-C_\infty\right)}{\left(C_0-C_\infty\right)x}+\frac{B_0\cos\alpha}{U_0}\sqrt{\frac{1}{\rho\mu_e}}H'(\eta)=0 \tag{2.12}$$

Calculation for equation (2.3):

$$H_x=\sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2x}}H(\eta)$$

Therefore, $u\dfrac{\partial H_x}{\partial x}=-\sqrt{\dfrac{\rho U_0\upsilon}{2\mu_e}}\dfrac{U_0 f'(\eta)}{2x^{\frac{3}{2}}}\left\{\eta H'(\eta)+H(\eta)\right\}$ (2.13)

Therefore, $v\dfrac{\partial H_x}{\partial y}=\dfrac{U_0}{2x}\sqrt{\dfrac{\upsilon U_0\rho}{2x\mu_e}}\left\{\eta f'(\eta)-f(\eta)\right\}H'(\eta)$ (2.14)

$$H_x\frac{\partial u}{\partial x}=-\frac{U_0\eta}{2x}\sqrt{\frac{U_0\rho\upsilon}{2\mu_e x}}H(\eta)f''(\eta) \tag{2.15}$$

$$\frac{B_0}{\mu_e}\cos\alpha\frac{\partial u}{\partial y}=\frac{B_0}{\mu_e}\cos\alpha\sqrt{\frac{U_0}{2\upsilon x}}\,U_0 f''(\eta) \tag{2.16}$$

$$\frac{\partial^2 H_x}{\partial y^2}=\frac{U_0}{2x}\sqrt{\frac{U_0\rho}{2\mu_e\upsilon x}}\;H''(\eta)$$

Therefore, $\dfrac{1}{\sigma\mu_e}\dfrac{\partial^2 H_x}{\partial y^2}=\dfrac{U_0}{2\sigma\mu_e x}\sqrt{\dfrac{U_0\rho}{2\mu_e\upsilon x}}\;H''(\eta)$ (2.17)

Now using equation (2.13)–(2.17) in equation (2.3) we get

$$-\sqrt{\frac{\rho U_0\upsilon}{2\mu_e}}\,\frac{U_0 f'(\eta)}{2x^{\frac{3}{2}}}\{\eta H'(\eta)+H(\eta)\}+\frac{U_0}{2x}\sqrt{\frac{\upsilon U_0\rho}{2x\mu_e}}\left[\eta f'(\eta)-f(\eta)\right]H'(\eta)$$

$$=-\frac{U_0\eta}{2x}\sqrt{\frac{U_0\rho\upsilon}{2\mu_e x}}H(\eta)\,f''(\eta)+\frac{B_0}{\mu_e}\cos\alpha\sqrt{\frac{U_0}{2\upsilon x}}\,U_0 f''(\eta)+\frac{U_0}{2\sigma\mu_e x}\sqrt{\frac{U_0\rho}{2\mu_e\upsilon x}}H''(\eta)$$

Dividing both sides by $\dfrac{U_0}{2\sigma\mu_e x^{\frac{3}{2}}}\sqrt{\dfrac{\rho U_0}{2\mu_e\upsilon}}$, we obtain

$$H''(\eta)+\mu_e\sigma\upsilon\{f(\eta)H'(\eta)+H(\eta)\,f'(\eta)\}-\mu_e\sigma\upsilon\eta H(\eta)\,f''(\eta)$$

$$+2\sqrt{\frac{1}{\rho\mu_e}}\,\frac{B_0\cos\alpha}{U_0}\,\mu_e\sigma\upsilon\;f''(\eta)=0 \tag{2.18}$$

<u>Calculation for equation (2.4)</u>:

$$\theta=\frac{T-T_\infty}{T_w-T_\infty}$$

$$T=T_\infty+(T_w-T_\infty)\,\theta$$

$$\frac{\partial T}{\partial x}=(T_w-T_\infty)\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}=(T_w-T_\infty)\theta'(\eta)\left(-\frac{\eta}{2x}\right)=-\frac{\eta}{2x}(T_w-T_\infty)\theta'(\eta)$$

Therefore, $u\dfrac{\partial T}{\partial x}=-\dfrac{\eta U_0}{2x}(T_w-T_\infty)\theta'(\eta)\,f'(\eta)$ (2.19)

$$\frac{\partial T}{\partial y}=(T_w-T_\infty)\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}=(T_w-T_\infty)\theta'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}=\sqrt{\frac{\upsilon_0}{2\gamma}}(T_w-T_\infty)\theta'(\eta)$$

Therefore, $v\dfrac{\partial T}{\partial y}=\sqrt{\dfrac{\upsilon U_0}{2x}}\sqrt{\dfrac{U_0}{2\upsilon x}}(T_w-T_\infty)\{\eta f'(\eta)-f(\eta)\}\theta'(\eta)$

$$=\frac{U_0}{2x}(T_w-T_\infty)\{\eta f'(\eta)-f(\eta)\}\theta'(\eta) \tag{2.20}$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{\partial}{\partial y}\left\{\frac{\partial T}{\partial y}\right\}=\frac{\partial}{\partial\eta}\left\{\frac{\partial T}{\partial y}\right\}\frac{\partial\eta}{\partial y}=\sqrt{\frac{U_0}{2\upsilon x}}\frac{\partial}{\partial\eta}\left\{\sqrt{\frac{U_0}{2\upsilon x}}(T_w-T_\infty)\theta'(\eta)\right\}=\frac{U_0}{2\upsilon x}(T_w-T_\infty)\theta''(\eta)$$

Therefore, $\dfrac{k}{\rho C}\dfrac{\partial^2 T}{\partial y^2}=\dfrac{U_0}{2\upsilon x}\dfrac{k}{\rho C_p}\left(T_w-T_\infty\right)\theta''(\eta)$ (2.21)

$$\frac{\upsilon}{Cp}\left(\frac{\partial u}{\partial y}\right)^2=\frac{U_0^3}{2C_p x}\left\{f''(\eta)\right\}^2 \tag{2.22}$$

$$\frac{1}{\rho C_p\sigma}\left\{\frac{\partial H_x}{\partial y}\right\}^2=\frac{U_0^2}{4C_p\sigma\mu_e x^2}\left\{H'(\eta)\right\}^2 \tag{2.23}$$

Now using equations (2.19)–(2.23) in equation (2.4) we get

$$-\frac{\eta U_0}{2x}\left(T_w-T_\infty\right)\theta'(\eta)\,f'(\eta)+\frac{U_0}{2x}\left(T_w-T_\infty\right)\left\{\eta f'(\eta)-f(\eta)\right\}\theta'(\eta)$$
$$=\frac{U_0}{2\upsilon x}\frac{k}{\rho C_p}\left(T_w-T_\infty\right)\theta''(\eta)+\frac{U_0^3}{2C_p x}\left\{f''(\eta)\right\}^2+\frac{U_0^2}{4C_p\sigma\mu_e x^2}\left\{H'(\eta)\right\}^2$$

Dividing both sides by $\dfrac{k}{\rho C_p}\left(T_w-T_\infty\right)\dfrac{U_0}{2\upsilon x}$, we obtain

$$-\frac{\rho C_p\upsilon}{k}\theta'(\eta)\,f(\eta)=\theta''(\eta)+\frac{U_0^2\upsilon\rho}{k\left(T_w-T_\infty\right)}\left\{f''(\eta)\right\}^2+\frac{U_0\rho\upsilon}{2\rho\mu_e xk\left(T_w-T_\infty\right)}\left\{H'(\eta)\right\}^2$$
$$+\frac{1}{2}\frac{pC_p\upsilon}{k}\frac{U_0^2}{C_p\left(T_w-T_\infty\right)}\frac{1}{\sigma\mu_e\upsilon}\left\{H'(\eta)\right\}^2=0 \tag{2.24}$$

<u>Calculation for equation (2.5)</u>:

$$\varphi=\frac{C-C_\infty}{\overline{x}\left(C_0-C_\infty\right)}\Rightarrow C=C_\infty+\frac{xU_0}{\upsilon}\varphi\left(C_0-C_\infty\right)$$

Therefore, $u\dfrac{\partial C}{\partial x}=\dfrac{U_0^2}{\upsilon}\left(C_0-C_\infty\right)\varphi(\eta)\,f'(\eta)-\dfrac{\eta U_0^2}{2\upsilon}\left(C_0-C_\infty\right)\varphi'(\eta)\,f'(\eta)$ (2.25)

and $v\dfrac{\partial C}{\partial y}=\dfrac{U_0^2}{2\upsilon}\left(C_0-C_\infty\right)\left[\eta f'(\eta)-f(\eta)\right]\varphi'(\eta)$ (2.26)

Therefore, $\dfrac{\partial^2 C}{\partial y^2}=\dfrac{U_0^2}{2\upsilon^2}\left(C_0-C_\infty\right)\varphi''(\eta)$ (2.27)

and $\dfrac{\partial^2 T}{\partial y^2}=\left(T_w-T_\infty\right)\dfrac{U_0}{2\upsilon x}\theta''(\eta)$ (2.28)

Now using equations (2.25)–(2.28) in equation (2.5) we get

$$\frac{U_0^2}{\upsilon}\left(C_0-C_\infty\right)\varphi(\eta)\,f'(\eta)-\frac{U_0^2}{2\upsilon}\left(C_0-C_\infty\right)\eta\varphi'(\eta)\,f'(\eta)+\frac{U_0^2}{2\upsilon}\left(C_0-C_\infty\right)\left\{\eta f'(\eta)-f(\eta)\right\}\varphi'(\eta)$$
$$=D_m\frac{U_0^2}{2\upsilon^2}\left(C_0-C_\infty\right)\varphi''(\eta)+D_T\frac{U_0}{2\upsilon x}\left(T_w-T_\infty\right)\theta''(\eta)$$

or, $\frac{U_0^2}{\upsilon}(C_0 - C_\infty)\varphi(\eta)\, f'(\eta) - \frac{U_0^2}{2\upsilon}(C_0 - C_\infty)\varphi'(\eta)\, f(\eta) = D_m \frac{U_0^2}{2\upsilon^2}(C_0 - C_\infty)\varphi''(\eta)$

$$+D_T(T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta)$$

Dividing both sides by $D_m \frac{U_0^2}{2\upsilon^2}(C_0 - C_\infty)$, we obtain

$$\varphi''(\eta) + \frac{\upsilon}{D_m} f(\eta)\varphi'(\eta) - \frac{\upsilon}{D_m} 2 f'(\eta)\varphi(\eta) + \frac{T_w - T_\infty}{C_0 - C_\infty}\frac{D_T}{xU_0}\frac{\upsilon}{D_m}\theta''(\eta) = 0 \tag{2.29}$$

<u>Boundary Conditions</u>:

For $y = 0$, we have $u = U_0$

But $y = 0 \implies \eta = 0$ and $u = U_0 \Rightarrow f'(\eta) = \frac{u}{U_0} = \frac{U_0}{U_0} = 1$

$$v = -V_0 \Rightarrow \sqrt{\frac{\upsilon U_0}{2x}}\left(\eta f'(\eta) - f(\eta)\right) = -V_0$$

or, $\eta f'(\eta) - f(\eta) = -V_0\sqrt{\frac{2x}{\upsilon U_0}}$

Since for $\eta = 0$, $f'(\eta) = 1$ and considering $f_w = V_0\sqrt{\frac{2x}{\upsilon U_0}}$, we obtain $f(0) = f_w$.

$$T = T_w \Rightarrow \theta(0) = \frac{T - T_\infty}{T_w - T_\infty} = \frac{T_w - T_\infty}{T_w - T_\infty} = 1$$

$$C = C_w \Rightarrow \varphi(0) = \frac{C - C_\infty}{\bar{x}(C_0 - C_\infty)} = \frac{C_w - C_\infty}{(C_0 - C_\infty)}\frac{C_0 - C_\infty}{(C_w - C_\infty)} = 1 \text{ as } \bar{x} = \frac{C_w - C_\infty}{(C_0 - C_\infty)}$$

Since $H_x = H_w$ when $y = 0$, hence

$$H_x = \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2x}}\, H(\eta) \Rightarrow \sqrt{\frac{U_0\rho\upsilon}{2x\mu_e}}\, H(\eta) = H_w \text{ when } \eta = 0$$

or, $H(\eta) = \sqrt{\frac{2x\mu_e}{U_0\rho\upsilon}}H_w$

or, $H(\eta) = \frac{\sqrt{2}H_w M}{H_0}$, where $M = \frac{B_0}{U_0}\sqrt{\frac{1}{\rho\mu_e}}$

$\therefore\ H(0) = h$, where $h = \frac{\sqrt{2}H_w M}{H_0}$.

Again $y \longrightarrow \infty \implies \eta \longrightarrow \infty$ and $u = 0$ as $y \to \infty \Rightarrow f'(\infty) = 0$

$$v = 0 \text{ as } y \to \infty \Rightarrow \sqrt{\frac{\upsilon U_0}{2x}}\left(\eta f'(\eta) - f(\eta)\right) = 0$$

or, $nf'(\eta) - f(\eta) = 0$

or, $f'(\eta) = \dfrac{f(\eta)}{n}$

Therefore, $f'(\infty) = 0$ as $\eta \to \infty$

$$T = T_w \Rightarrow \theta(0) = \frac{T - T_\infty}{T_w - T_\infty} = \frac{T_\infty - T_\infty}{T_w - T_\infty} = 0$$

$$C = C_\infty \Rightarrow \varphi(\infty) = \frac{C - C_\infty}{\overline{x}(C_0 - C_\infty)} = \frac{C_\infty - C_\infty}{\overline{x}(C_0 - C_\infty)} = 0$$

$$H_x = 0 \text{ as } y \to \infty \Rightarrow 0 = \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0 \upsilon}{2x}}\, H(\eta) \text{ as } \eta \to \infty \text{ or, } H(\infty) = 0$$

Now we define the following dimensionless numbers and parameters

The local Grashoff number: $G_r = \dfrac{g\beta(T_w - T_\infty)2x}{U_0^2}$

The local modified Grashoff number: $G_m = \dfrac{g\beta^*(C_0 - C_\infty)2x^2}{U_0^2}$

The magnetic parameter: $M = \dfrac{B_0}{U_0}\sqrt{\dfrac{1}{\rho\mu_e}}$

The magnetic diffusivity parameter: $P_m = \sigma\upsilon\mu_e$

The Prandtl number: $P_r = \dfrac{\rho\upsilon C_p}{k}$

The Eckert number: $E_c = \dfrac{U_0^2}{C_p(T_w - T_\infty)}$

The Schmidt number: $S_c = \dfrac{\upsilon}{D_m}$

The Soret number: $S_0 = \dfrac{(T_w - T_\infty)D_T}{(C_0 - C_\infty)U_0 x}$ and

The suction parameter $f_w = V_0\sqrt{\dfrac{2x}{\upsilon U_0}}$.

Therefore, introducing the above dimensionless numbers and parameters into equations (2.12), (2.18), (2.24) and (2.29), we obtain

$$f'''(\eta) + f(\eta) f''(\eta) + G_r\theta + G_m\varphi + M\cos\alpha\, H'(\eta) = 0 \tag{2.30}$$

$$H''(\eta) + P_m\{f(\eta)H'(\eta) + H(\eta) f'(\eta)\} - P_m\,\eta H(\eta) f''(\eta) + 2P_m M\cos\alpha\, f''(\eta) = 0 \tag{2.31}$$

$$\theta''(\eta) + P_r\theta'(\eta) f(\eta) + P_r E_c\left[\{f''(\eta)\}^2 + \frac{1}{2P_m}\{H'(\eta)\}^2\right] = 0 \tag{2.32}$$

$$\varphi''(\eta) + S_c f(\eta)\varphi'(\eta) - 2S_c f'(\eta)\varphi(\eta) + S_0 S_c\theta''(\eta) = 0 \tag{2.33}$$

with transformed boundary conditions

$$\left.\begin{aligned}&f(0)=f_w,\ f'(0)=1,\theta(0)=1,\varphi(0)=1,H(0)=h=1\\&f'(\infty)=0,\theta(\infty)=0,\varphi(\infty)=0,H(\infty)=0\end{aligned}\right\}\tag{2.34}$$

Where θ is the dimensionless temperature and it is assumed that $h=\dfrac{\sqrt{2}H_wM}{H_0}=1$ and $\dfrac{U_0x}{\upsilon}=1$.

To obtain the solution for large suction of above equations (2.30)–(2.33) we further introduce the following transformations:

$$\xi=\eta f_w,\ f(\eta)=f_wF(\xi),H(\eta)=f_wL(\xi),\theta(\eta)=f_w^2G(\xi),\ \text{and}\,\varphi(\eta)=f_w^2P(\xi)\tag{2.35}$$

<u>Calculation for equation (2.30)</u>:

$$\xi=\eta f_w\Rightarrow\frac{\partial\xi}{\partial\eta}=f_w$$

$$f(\eta)=f_wF(\xi)$$

$$f'(\eta)=f_wF'(\xi)\frac{\partial\xi}{\partial\eta}=f_w^2F'(\xi)$$

$$f''(\eta)=f_w^2F''(\xi)\frac{\partial\xi}{\partial\eta}=f_w^3F''(\xi)$$

$$f'''(\eta)=f_w^3F'''(\xi)\frac{\partial\xi}{\partial\eta}=f_w^4F'''(\xi)\tag{2.36}$$

$$f(\eta)f''(\eta)=f_wF(\xi)f_w^3F''(\xi)=f_w^4F(\xi)F''(\xi)\tag{2.37}$$

$$G_r\theta(\eta)=G_rf_w^2G(\xi)\tag{2.38}$$

$$G_m\varphi(\eta)=G_mf_w^2P(\xi)\tag{2.39}$$

$$M\cos\alpha\,H'(\eta)=M\cos\alpha\,f_wL'(\xi)\frac{\partial\xi}{\partial\eta}=M\cos\alpha\ f_w^2L'(\xi)\tag{2.40}$$

Now using equations (2.36)–(2.40) in equation (2.30) we get

$$f_w^4F'''(\xi)+f_w^4F(\xi)F''(\xi)+G_rf_w^2G(\xi)+G_mf_w^2P(\xi)+f_w^2L'(\xi)M\cos\alpha=0$$

or, $$F'''(\xi)+F(\xi)F''(\xi)+\frac{1}{f_w^2}\left[G_rG(\xi)+G_mP(\xi)+L'(\xi)M\cos\alpha\right]=0\tag{2.41}$$

<u>Calculation for equation (2. 31)</u>:

$$H(\eta)=f_wL(\xi)\Rightarrow H'(\eta)=f_wL'(\xi)\frac{\partial\xi}{\partial\eta}=f_w^2L'(\xi)$$

$$H''(\eta) = f_w^2 L''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^3 L''(\xi) \tag{2.42}$$

$$f'(\eta)H(\eta) = f_w^2 F'(\xi)\, f_w L(\xi) = f_w^3 F'(\xi)L(\xi) \tag{2.43}$$

$$f(\eta)H'(\eta) = f_w F(\xi)\, f_w^2 L'(\xi) = f_w^3 F(\xi)L'(\xi) \tag{2.44}$$

$$H(\eta)\, f''(\eta) = f_w L(\xi)\, f_w^3 F''(\xi) = f_w^4 L(\xi)F''(\xi) \tag{2.45}$$

$$f''(\eta) = f_w^3 F''(\xi) \tag{2.46}$$

Now using equations (2.42)–(2.46) in equation (2.31) we get

$$f_w^3 L''(\xi) + P_m\left[f_w^3 F'(\xi)L(\xi) + f_w^3 F(\xi)L'(\xi)\right] - P_m \xi f_w^3 L(\xi)F''(\xi)$$
$$+2P_m f_w^3 F''(\xi) M\cos\alpha = 0$$

or, $L''(\xi) + P_m\left[F'(\xi)L(\xi) + F(\xi)L'(\xi)\right] - P_m \xi L(\xi)F''(\xi) + 2P_m F''(\xi) M\cos\alpha = 0$ (2.47)

<u>Calculation for equation (2.32)</u>:

$$\theta(\eta) = f_w^2 G(\xi) \Rightarrow \theta'(\eta) = f_w^2 G'(\xi)\frac{\partial \xi}{\partial \eta} = f_w^3 G'(\xi)$$

$$\theta''(\eta) = f_w^3 G''\frac{\partial \xi}{\partial \eta} = f_w^4 G''(\xi) \tag{2.48}$$

$$f(\eta)\theta'(\eta) = f_w F(\xi)\, f_w^3 G'(\xi) = f_w^4 F(\xi)G'(\xi) \tag{2.49}$$

$$\left(f''(\eta)\right)^2 = f_w^6\left(F''(\xi)\right)^2 \tag{2.50}$$

$$\left(H'(\eta)\right)^2 = f_w^4\left(L'(\xi)\right)^2 \tag{2.51}$$

Now using equations (2.48)–(2.51) in equation (2.32) we get

$$f_w^4 G''(\xi) + P_r f_w^4 F(\xi)G'(\xi) + P_r E_c\left[f_w^6\left(F''(\xi)\right)^2 + \frac{1}{2P_m} f_w^4\left(L'(\xi)\right)^2\right] = 0$$

or, $G''(\xi) + P_r F(\xi)G'(\xi) + P_r E_c\left[f_w^2\left(F''(\xi)\right)^2 + \dfrac{1}{2P_m}\left(L'(\xi)\right)^2\right] = 0$ (2.52)

<u>Calculation for equation (2.33)</u>:

$$\varphi(\eta) = f_w^2 P(\xi) \Rightarrow \varphi'(\eta) = f_w^2 P'(\xi)\frac{\partial \xi}{\partial \eta} = f_w^3 P'(\xi)$$

$$\varphi''(\eta) = f_w^3 P''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^4 P''(\xi) \tag{2.53}$$

$$f'(\eta)\varphi(\eta) = f_w^2 F'(\xi)\, f_w^2 P(\xi) = f_w^4 F'(\xi)P(\xi) \tag{2.54}$$

and $f(\eta)\varphi'(\eta) = f_w^3 F(\xi)\, f_w P'(\xi) = f_w^4 F(\xi)P'(\xi)$

$$\theta''(\eta) = f_w^3 G''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^4 G''(\xi) \tag{2.55}$$

Now using equations (2.53)–(2.55) in equation (2.33) we get

$$f_w^4 P''(\xi) - 2S_c f_w^4 F'(\xi)P(\xi) + S_c f_w^4 F(\xi)P'(\xi) + S_0 S_c f_w^4 G''(\xi) = 0$$

or, $P''(\xi) - 2S_c F'(\xi)P(\xi) + S_c F(\xi)P'(\xi) + S_0 S_c G''(\xi) = 0$ (2.56)

<u>Boundary Conditions (2.34)</u>:

For $\eta = 0 \Rightarrow \xi = 0$ and then $f(\eta) = f_w \Rightarrow f(\eta) = f_w F(\xi)$ or, $f_w = f_w F(\xi) \Rightarrow F(\xi) = 1$

$$f'(\eta) = 1 \Rightarrow f_w^2 F'(\xi) = 1 \text{ or, } F'(\xi) = \frac{1}{f_w^2} = \varepsilon$$

$$\theta(\eta) = 1 \Rightarrow f_w^2 G(\xi) = 1 \text{ or, } G(\xi) = \frac{1}{f_w^2} = \varepsilon$$

$$\varphi(\eta) = 1 \Rightarrow f_w^2 P(\xi) = 1 \text{ or, } P(\xi) = \frac{1}{f_w^2} = \varepsilon$$

$$H(\eta) = 1 \Rightarrow f_w L(\xi) = 1 \text{ or, } L(\xi) = \frac{1}{f_w} = \sqrt{\varepsilon}$$

For $\xi \to \infty$

$$f'(\eta) = 0 \Rightarrow F'(\xi) = 0$$

$$\theta(\eta) = 0 \Rightarrow G(\xi) = 0$$

$$\varphi(\eta) = 0 \Rightarrow P(\xi) = 0$$

$$H(\eta) = 0 \Rightarrow L(\xi) = 0$$

Therefore, re-writing equations (2.41), (2.47), (2.52), (2.56) and the corresponding boundary conditions:

$$F'''(\xi) + F(\xi)F''(\xi) + \varepsilon\left[G_r G(\xi) + G_m P(\xi) + M\cos\alpha\, L'(\xi)\right] = 0 \tag{2.57}$$

$$L''(\xi) + P_m\left[F'(\xi)L(\xi) + F(\xi)L'(\xi)\right] - P_m \xi L(\xi)F''(\xi) + 2P_m M\cos\alpha F''(\xi) = 0 \tag{2.58}$$

$$G''(\xi) + P_r F(\xi)G'(\xi) + P_r E_c\left[\frac{1}{\varepsilon}\left(F''(\xi)\right)^2 + \frac{1}{2P_m}\left(L'(\xi)\right)^2\right] = 0 \tag{2.59}$$

$$P''(\xi) - 2S_c F'(\xi)P(\xi) + S_c F(\xi)P'(\xi) + S_0 S_c G''(\xi) = 0 \tag{2.60}$$

with transformed boundary conditions

$$\left.\begin{aligned} &F(\xi) = 1, F'(\xi) = \varepsilon, G(\xi) = \varepsilon, P(\xi) = \varepsilon\ L(\xi) = \sqrt{\varepsilon} \text{ at } \xi = 0 \\ &F'(\xi) = 0, G(\xi) = 0, P(\xi) = 0\ L(\xi) = 0 \text{ as } \xi \to \infty \end{aligned}\right\} \tag{2.61}$$

CHAPTER III

Perturbation Solution

To obtain a complete solution of the coupled nonlinear system of equations (2.57) – (2.60) under boundary conditions (2.61), we introduce the perturbation approximation technique. Since the dependent variables F, L, G and P are mostly dependent on ξ only and the fluid is purely incompressible, we expand the dependent variables F, L, G and P in powers of small perturbation quantity ε such that the terms in ε^3 and its higher order can be neglected. Thus we assume

$$F(\xi)=1+\varepsilon F_1(\xi)+\varepsilon^2 F_2(\xi)+\varepsilon^3 F_3(\xi)+\ldots\ldots\ldots \tag{3.1}$$

$$L(\xi)=\varepsilon L_1(\xi)+\varepsilon^2 L_2(\xi)+\varepsilon^3 L_3(\xi)+\ldots\ldots\ldots \tag{3.2}$$

$$G(\xi)=\varepsilon G_1(\xi)+\varepsilon^2 G_2(\xi)+\varepsilon^3 G_3(\xi)+\ldots\ldots\ldots \tag{3.3}$$

$$P(\xi)=\varepsilon P_1(\xi)+\varepsilon^2 P_2(\xi)+\varepsilon^3 P_3(\xi)+\ldots\ldots\ldots \tag{3.4}$$

By differentiating equation (3.1):

Therefore, $F^{/}(\xi)=\varepsilon F_1^{/}+\varepsilon^2 F_2^{/}+\varepsilon^3 F_3^{/}+\ldots\ldots\ldots$

$$F^{//}(\xi)=\varepsilon F_1^{//}+\varepsilon^2 F_2^{//}+\varepsilon^3 F_3^{//}+\ldots\ldots\ldots$$

$$F^{///}(\xi)=\varepsilon F_1^{///}+\varepsilon^2 F_2^{///}+\varepsilon^3 F_3^{///}+\ldots\ldots\ldots$$

<u>Calculation for equation (2.57)</u>:

$$F'''(\xi)+F(\xi)F''(\xi)+\varepsilon\left[G_r G(\xi)+G_m P(\xi)+M\cos\alpha\, L'(\xi)\right]=0$$

$$\varepsilon F_1^{///}+\varepsilon^2 F_2^{///}+\varepsilon^3 F_3^{///}+\ldots\ldots+\left(1+\varepsilon F_1+\varepsilon^2 F_2+\varepsilon^3 F_3+\ldots\right)\left(\varepsilon F_1^{//}+\varepsilon^2 F_2^{//}+\varepsilon^3 F_3^{//}+\ldots\right)$$
$$+\varepsilon\Big[G_r\left(\varepsilon G_1+\varepsilon^2 G_2+\varepsilon^3 G_3+\ldots\right)+G_m\left(\varepsilon P_1+\varepsilon^2 P_2+\varepsilon^3 P_3+\ldots\right)$$
$$+M\cos\alpha\left(\varepsilon L_1^{/}+\varepsilon^2 L_2^{/}+\varepsilon^3 L_3^{/}+\ldots\right)\Big]=0$$

The first order perturbation equation $O(\varepsilon)$: (Those terms including the order of ε)

$$\varepsilon F_1^{///}+\varepsilon F_1^{//}=0\Rightarrow F_1^{///}+F_1^{//}=0$$

The second order perturbation equation $O(\varepsilon^2)$: (Those terms including the order of ε^2)

$$F_2''' + F_1F_1''+F_2''+G_rG_1+G_mP_1+M\cos\alpha\, L_1'=0$$

By differentiating equation (3.2):

$$L'=\varepsilon L_1'+\varepsilon^2 L_2{}'+\varepsilon^3 L_3'+\ldots\ldots$$

$$L''=\varepsilon L_1''+\varepsilon L_2''+\varepsilon^3 L_3''+\ldots\ldots$$

Calculation for equation (2.58):

$$L''(\xi)+P_m\left[F'(\xi)L(\xi)+F(\xi)L'(\xi)\right]-P_m\xi L(\xi)F''(\xi)+2P_mM\cos\alpha F''(\xi)=0$$

$$\varepsilon L_1''+\varepsilon^2L_2''+\varepsilon^3L_3''+\ldots+P_m\left(\varepsilon F_1'+\varepsilon^2F_2'+\varepsilon^3F_3'+\ldots\right)\left(\varepsilon L_1+\varepsilon^2L_2+\varepsilon^3L_3+\ldots\right)$$

$$+P_m\left(1+\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3+\ldots\right)\left(\varepsilon L_1'+\varepsilon^2L_2'+\varepsilon^3L_3'+\ldots\right)$$

$$-P_m\xi\left(\varepsilon L_1+\varepsilon^2L_2+\varepsilon^3L_3+\ldots\right)\left(\varepsilon F_1^{//}+\varepsilon^2F_2^{//}+\varepsilon^3F_3^{//}+...\right)$$

$$+2P_mM\cos\alpha\left(\varepsilon F_1''+\varepsilon^2F_2''+\varepsilon^3F_3''+\ldots\right)=0$$

The first order perturbation equation $O(\varepsilon)$:

$$L_1''+P_mL_1'+2P_mM\cos\alpha F_1''=0$$

The second order perturbation equation $O(\varepsilon^2)$:

$$L_2''+P_m\left(F_1'L_1+F_1L_1'+L_2'\right)-P_m\xi L_1F_1''+2P_mF_2''M\cos\alpha=0$$

By differentiating equation (3.3):

$$G'=\varepsilon G_1'+\varepsilon^2G_2'+\varepsilon^3G_3'+\ldots\ldots$$

$$G''=\varepsilon G_1''+\varepsilon^2G_2''+\varepsilon^3G_3''+\ldots\ldots$$

Calculation for equation (2.59):

$$G''(\xi)+P_rF(\xi)G'(\xi)+P_rE_c\left[\frac{1}{\varepsilon}\left(F''(\xi)\right)^2+\frac{1}{2P_m}\left(L'(\xi)\right)^2\right]=0$$

$$\varepsilon G_1''+\varepsilon^2G_2''+\varepsilon^3G_3''+\ldots+P_r\left(1+\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3+\ldots\right)\left(\varepsilon G_1'+\varepsilon^2G_2'+\varepsilon^3G_3'+\ldots\right)$$

$$+P_rE_c\left\{\frac{1}{\varepsilon}\left(\varepsilon F_1''+\varepsilon^2F_2''+\varepsilon^3F_3''+\ldots\right)+\frac{1}{2P_m\left(\varepsilon L_1'+\varepsilon^2L_2'+\varepsilon^3L_3'+\ldots\right)^2}\right\}=0$$

The first order perturbation equation $O(\varepsilon)$:

$$G_1''+P_rG_1'=0$$

The second order perturbation equation $O(\varepsilon^2)$:

$$G_2''+P_r\left(G_2''+F_1G_1'\right)+\frac{E_cP_r}{2P_m}\left(L_1'\right)^2=0$$

By differentiating equation (3.4):

$$P'=\varepsilon P_1'+\varepsilon^2P_2'+\varepsilon^3P_3'+\ldots$$

$$P''=\varepsilon P_1''+\varepsilon^2P_2''+\varepsilon^3P_3''+\ldots$$

Calculation for equation (2.60):

$$P''(\xi)-2S_cF'(\xi)P(\xi)+S_cF(\xi)P'(\xi)+S_0S_cG''(\xi)=0$$

$$\varepsilon P_1''+\varepsilon^2P_2''+\varepsilon^3P_3''+\ldots-2S_c\left(\varepsilon F_1'+\varepsilon^2F_2'+\varepsilon^3F_3'+\ldots\right)\left(\varepsilon P_1+\varepsilon^2P_2+\varepsilon^3P_3+\ldots\ldots\right)$$

$$+S_c\left(\varepsilon P_1'+\varepsilon^2P_2'+\varepsilon^3P_3'+\ldots\right)\left(1+\varepsilon F_1+\varepsilon^2F_2+\varepsilon^3F_3+\ldots\right)$$

$$+S_oS_c\left(\varepsilon G_1''+\varepsilon^2G_2''+\varepsilon^3G_3''+\ldots\right)=0$$

The first order perturbation equation $O(\varepsilon)$: (Those terms including the order of ε)

$$P_1''+S_cP_1'+S_oS_cG_1''=0$$

The second order perturbation equation $O(\varepsilon^2)$: (Those terms including the order of ε^2)

$$P_2''-2S_cP_1F_1'+S_c\left(P_1'F_1+P_2'\right)+S_oS_cG_2''=0$$

So that the first order equations are:

$$F_1'''+F_1''=0 \tag{3.5}$$

$$L_1''+P_mL_1'+2P_mM\cos\alpha F_1''=0 \tag{3.6}$$

$$G_1''+P_rG_1'=0 \tag{3.7}$$

$$P_1''+S_cP_1'+S_oS_cG_1''=0 \tag{3.8}$$

The corresponding boundary conditions are:

$$\left.\begin{aligned}&F_1=0,\ F_1'=1,\ G_1=1,\ P_1=1,\ L_1=\frac{1}{\sqrt{\varepsilon}}\ \text{at}\ \xi=0\\&F_1'=0,\ G_1=0,\ P_1=0,\ L_1=0,\ \text{as}\ \xi\to\infty\end{aligned}\right\} \tag{3.9}$$

and the second order equations are:

$$F_2'''+F_1F_1''+F_2''+G_rG_1+G_mP_1+L_1'M\cos\alpha=0 \tag{3.10}$$

$$L_2''+P_m\left(F_1'L_1+F_1L_1'+L_2'\right)-P_m\xi L_1F_1''+2P_mF''M\cos\alpha=0 \tag{3.11}$$

$$G_2''+P_r\left(G_2'+F_1G_1'\right)+\frac{E_cP_r}{2P_m}\left(L_1'\right)^2=0 \tag{3.12}$$

$$P_2''-2S_cP_1F_1'+S_c\left(P_1'F_1+P_2'\right)+S_oS_cG_2''=0 \tag{3.13}$$

with the corresponding boundary conditions:

$$\left.\begin{aligned}&F_2=0,\ F_2'=0,\ G_2=0,\ P_2=0,\ L_2=0\ \text{at}\ \xi=0\\&F_2'=0,\ G_2=0,\ P_2=0,\ L_2=0\ \text{as}\ \xi\to\infty\end{aligned}\right\} \tag{3.14}$$

Now we are interested to solve equations (3.5)–(3.8) with boundary conditions (3.9) and equations (3.10)–(3.13) with boundary conditions (3.14)

From equation (3.5) we have

$F_1''' + F_1'' = 0$

The general solution of equation (3.5) is given by

$F_1 = c_1 + c_2\xi + c_3 e^{-\xi}$

Applying boundary conditions:

$F_1 = 0,\ F_1' = 1$ as $\xi = 0$ and $F_1' = 0$ at $\xi \to \infty$

Therefore $0 = c_1 + c_3,\ F_1' = c_2 - c_3 e^{-\xi},\ 1 = c_2 - c_3$ and $0 = c_2$ so that $c_1 = 1,\ c_2 = 0,\ c_3 = -1$

Hence the complete solution of equation (3.5) $F_1(\xi) = 1 - e^{-\xi}$ (3.15)

Again from equation (3.6) we have

$L_1'' + P_m L_1' + 2P_m M\cos\alpha\, F_1'' = 0$

Here $F_1 = 1 - e^{-\xi}$ Therefore $F_1'' = -e^{-\xi}$

So that $L_1'' + P_m L_1' = 2P_m M\cos\alpha\, e^{-\xi}$

The complementary function is obtained by

$L_{1c} = c_1 + c_2 e^{-p_m\xi}$

Now the particular integral

$$L_{1p} = \frac{1}{D^2 + P_m D} 2P_m M\cos\alpha\, e^{-\xi}$$

$$= \frac{2P_m M\cos\alpha\, e^{-\xi}}{1 - P_m}$$

The general solution is $L_1 = c_1 + c_2 e^{-P_m\xi} + \dfrac{2P_m M\cos\alpha\, e^{-\xi}}{1 - P_m}$

Using boundary conditions:

$L_1 = \dfrac{1}{\sqrt{\varepsilon}}$ at $\xi = 0$ and $L_1 = 0$ as $\xi \to \infty$

Therefore $\dfrac{1}{\sqrt{\varepsilon}} = c_1 + c_2 + \dfrac{2M\ P_m\cos\alpha}{1 - P_m}$ and $c_1 = 0,$ so that $c_1 = 0,\ c_2 = \dfrac{1}{\sqrt{\varepsilon}} - \dfrac{2M\ P_m\cos\alpha}{1 - P_m}$

Hence the complete solution of equation (3.6) is

$$L_1 = \frac{2P_m M\cos\alpha}{1 - P_m} e^{-\xi} + \left(\frac{1}{\sqrt{\varepsilon}} - \frac{2P_m M\cos\alpha}{1 - P_m}\right) e^{-P_m\xi}$$

or, $L_1 = \dfrac{2P_m M\cos\alpha}{1 - P_m}\left(e^{-\xi} - e^{-P_m\xi}\right) + \dfrac{1}{\sqrt{\varepsilon}} e^{-P_m\xi}$

or, $L_1(\xi) = A_1\left(e^{-\xi} - e^{P_m\xi}\right) + K_1 e^{-P_m\xi}$

where $A_1 = \dfrac{2P_m M\cos\alpha}{1-P_m}$, $K_1 = \dfrac{1}{\sqrt{M\cos\alpha}}$ (3.16)

Again from equation (3.7) we have

$$G_1'' + P_r G_1' = 0$$

The general solution is $G_1 = c_1 + c_2 e^{-P_r\xi}$

Using boundary conditions:

$G_1 = 1$ at $\xi = 0$ and $G_1 = 0$ as $\xi \to \infty$

Therefore, $1 = c_1 + c_2$ and $0 = c_1$ so that $c_1 = 0, c_2 = 1$

Hence the complete solution of equation (3.7) is

$$G_1(\xi) = e^{-P_r\xi} \tag{3.17}$$

Again from equation (3.8) we have

$$P_1'' + S_c P_1' + S_o S_c G_1'' = 0$$

Here $G_1 = e^{-P_r\xi}$ Therefore $G_1' = -P_r e^{-P_r\xi}$, $G_1'' = P_r^2 e^{-P_r\xi}$

Therefore $P_1'' + S_c P_1' = -S_o S_c P_r^2 e^{-P_r\xi}$

The complementary function is obtained by

$$P_{1c} = c_1 + c_2 e^{-S_c\xi}$$

And the particular integral is

$$P_{1p} = \frac{1}{D^2 + S_c D}\left(-S_0 S_c P_r^2 e^{-P_r\xi}\right)$$

or, $P_{1p} = \dfrac{-S_0 S_c P_r^2 e^{-P_r\xi}}{P_r^2 - S_c P_r} = -\dfrac{S_0 S_c P_r}{P_r - S_c} e^{-P_r\xi}$

The general solution is $P_1 = c_1 + c_2 e^{-S_c\xi} - \dfrac{S_o S_c P_r e^{-P_r\xi}}{P_r - S_c}$

Using boundary conditions:

$P_1 = 1$ at $\xi = 0$ and $P_1 = 0$ as $\xi \to \infty$

Therefore, $1 = c_1 + c_2 - \dfrac{S_o S_c P_r}{P_r - S_c}$ and $0 = c_1$ so that $c_1 = 0, c_2 = 1 + \dfrac{S_o S_c P_r}{P_r - S_c}$

Hence the complete solution of equation (3.8) is

$$P_1 = \left(1 + \frac{S_o S_c P_r}{P_r - S_c}\right) e^{-S_c\xi} - \frac{S_o S_c P_r}{P_r - S_c} e^{-P_r\xi}$$

or, $P_1(\xi) = A_2 e^{-S_c\xi} + A_3 e^{-P_r\xi}$ (3.18) where, $A_2 = 1 - A_3,\ A_3 = -\dfrac{S_o S_c P_r}{P_r - S_c}$

Again from equation (3.10) we have

$$F_2''' + F_1 F_1'' + F_2'' + G_r G_1 + G_m P_1 + M\cos\alpha\, L_1' = 0$$

Here $F_1 = 1 - e^{-\xi},\ F_1'' = -e^{-\xi},\ G_1 = e^{-P_r\xi},\ P_1 = A_2 e^{-S_e\xi} + A_3 e^{-P_r\xi}$,

$L_1 = A_1\left(e^{-\xi} - e^{P_m\xi}\right) + K_1 e^{-P_m\xi}$,

$L_1' = A_1\left(P_m e^{-P_m\xi} - e^{-\xi}\right) - K_1 P_m e^{-P_m\xi}$

Therefore, $F_2''' + F_2'' = \left(1 - e^{-\xi}\right)e^{-\xi} - G_r e^{-P_r\xi} - G_m A_2 e^{-S_c\xi} - G_m A_3 e^{-P_r\xi} - A_1 M\cos\alpha\left(P_m e^{-P_m\xi} - e^{-\xi}\right)$

$+ K_1 P_m M\cos\alpha\, e^{-P_m\xi}$

The complementary function is obtained by

$$F_{2c} = c_1 + c_2\xi + c_3 e^{-\xi}$$

and the particular integral is

$$F_{2p} = \xi e^{-\xi} + \frac{1}{4}e^{-2\xi} + \frac{G_r}{P_r^2\left(P_r - 1\right)}e^{-P_r\xi} + \frac{G_m A_2}{S_c^2\left(S_c - 1\right)}e^{-S_c\xi} + \frac{G_m A_3}{P_r^2\left(P_r - 1\right)}e^{-P_r\xi} + \frac{A_1 M\cos\alpha}{P_m\left(P_m - 1\right)}e^{-P_m\xi}$$
$$+ A_1 M\cos\alpha\,\xi e^{-\xi} - \frac{K_1 M\cos\alpha}{P_m\left(P_m - 1\right)}e^{-P_m\xi}$$

The general solution is

$$F_2 = c_1 + c_2\xi + c_3 e^{-\xi} + \frac{1}{4}e^{-2\xi} + \left(1 + A_1 M\cos\alpha\right)\xi e^{-\xi} + \frac{\left(G_r + G_m A_3\right)}{P_r^2\left(P_r - 1\right)}e^{-P_r\xi}$$
$$+ \frac{G_m A_2}{S_c^2\left(S_c - 1\right)}e^{-S_c\xi} + \frac{\left(A_1 - K_1\right)M\cos\alpha}{P_m\left(P_m - 1\right)}e^{-P_m\xi}$$

Using boundary conditions:

$F_2 = 0,\ F_2' = 0$ as $\xi = 0$ and $F_2' = 0$ as $\xi \to \infty$

$$c_1 = \frac{1}{2} - \left(1 + A_1 M\cos\alpha\right) + \frac{\left(G_r + G_m A_3\right)}{P_r\left(P_r - 1\right)} + \frac{G_m A_2}{S_c\left(S_c - 1\right)} + \frac{\left(A_1 - K_1\right)M\cos\alpha}{\left(P_m - 1\right)}$$
$$- \frac{1}{4} - \frac{\left(G_r + G_m A_3\right)}{P_r^2\left(P_r - 1\right)} - \frac{G_m A_2}{S_c^2\left(S_c - 1\right)} - \frac{\left(A_1 - K_1\right)M\cos\alpha}{\left(P_m - 1\right)}$$

$$c_2 = 0$$

$$c_3 = -\frac{1}{2} + \left(1 + A_1 M\cos\alpha\right) - \frac{\left(G_r + G_m A_3\right)}{P_r\left(P_r - 1\right)} - \frac{G_m A_2}{S_c\left(S_c - 1\right)} - \frac{\left(A_1 - K_1\right)M\cos\alpha}{\left(P_m - 1\right)}$$

$$+\left(1+A_1M\cos\alpha\right)\xi e^{-\xi}+\frac{\left(G_r+G_mA_3\right)}{P_r^2\left(P_r-1\right)}e^{-P_r\xi}+\frac{G_mA_2}{S_c^2\left(S_c-1\right)}e^{-S_c\xi}+\frac{\left(A_1-K_1\right)M\cos\alpha}{P_m\left(P_m-1\right)}e^{-P_m\xi}$$

Hence the complete solution of equation (3.10) is

$$F_2\left(\xi\right)=\frac{1}{4}e^{-2\xi}+D_1\xi e^{-\xi}+D_9e^{-P_r\xi}+D_{10}e^{-S_c\xi}+D_7e^{-P_m\xi}+D_8e^{-\xi}+D_{11} \tag{3.19}$$

Where $D_1=1+A_1M\cos\alpha$, $D_2=G_r+G_mA_3$, $D_3=G_mA_2$, $D_4=\dfrac{D_2}{P_r\left(P_r-1\right)}$,

$$D_5=\frac{D_3}{S_c\left(S_c-1\right)},\ D_6=\frac{D_{12}}{P_m-1}$$

$$D_{12}=\left(A_1-K_1\right)M\cos\alpha,\ D_7=\frac{D_{12}}{P_m\left(P_m-1\right)},\ D_8=-\frac{1}{2}+D_1-D_4-D_5-D_6,$$

$$D_9=\frac{D_2}{P_r^2\left(P_r-1\right)},\ D_{10}=\frac{D_3}{S_c^2\left(S_c-1\right)},\ D_{11}=-\frac{1}{4}-D_8-D_9-D_{10}-D_7$$

Again from equation (3.11) we have

$$L_2''+P_mL_2'+P_m\left(F_1'L_1+F_1L_1'\right)-P_m\xi L_1F_1''+2P_mM\cos\alpha F_2''=0$$

Here $F_1=1-e^{-\xi}$, $F_1'=e^{-\xi}$, $F_1''=-e^{-\xi}$

$$F_2=\frac{1}{4}e^{-2\xi}+D_1\xi e^{-\xi}+D_9e^{-P_r\xi}+D_{10}e^{-S_c\xi}+D_7e^{-P_m\xi}+D_8e^{-\xi}+D_{11}$$

$$F_2'=-\frac{1}{2}e^{-2\xi}-D_1\xi e^{-\xi}+D_1e^{-\xi}-D_9P_re^{-P_r\xi}-D_{10}S_ce^{-S_c\xi}-D_7P_me^{-P_m\xi}-D_8e^{-\xi}$$

$$F_2''=e^{-2\xi}+D_1\xi e^{-\xi}+\left(D_8-2D_1\right)e^{-\xi}+D_9P_r^2e^{-P_r\xi}+D_{10}S_c^2e^{-S_c\xi}+D_7P_m^2e^{-P_m\xi}$$

$$L_1=A_1\left(e^{-\xi}-e^{-P_m\xi}\right)+K_1e^{-P_m\xi},\ L_1'=A_1\left(P_me^{-P_m\xi}-e^{-\xi}\right)-K_1P_me^{-P_m\xi}$$

$$L_2''+P_mL_2'+P_m\left(K_1-A_1-A_1P_m+K_1P_m\right)e^{-\left(P_m+1\right)}+P_m^2\left(A_1-K_1\right)e^{P_m\xi}+A_1P_me^{-\xi}+A_1P_m\xi e^{-2\xi}$$
$$+\left(K_1-A_1\right)\xi e^{-\left(P_m+1\right)\xi}+2P_mM\cos\alpha\, e^{-2\xi}+2P_m\left(D_8-2D_1\right)M\cos\alpha\, e^{-\xi}+2P_mD_1M\cos\alpha\,\xi e^{-\xi}$$
$$+2P_mD_9P_r^2M\cos\alpha\, e^{-P_r\xi}+2P_mD_{10}S_c^2M\cos\alpha\, e^{-S_c\xi}+2P_m^3D_7M\cos\alpha\, e^{-P_m\xi}=0$$

The complementary function is obtained by

$$L_{2c}=c_1+c_2e^{-P_m\xi}$$

Now the particular integral is

$$L_{2p}=P_m\left(A_1-K_1\right)e^{-\left(P_m+1\right)\xi}+P_m\left(A_1-K_1\right)\xi e^{-P_m\xi}-\frac{A_1P_m}{1-P_m}e^{-\xi}+\frac{A_1P_m\xi}{2\left(P_m-2\right)}e^{-2\xi}$$
$$+\frac{A_1P_m\left(P_m-4\right)}{4\left(P_m-2\right)^2}e^{-2\xi}+\frac{\left(A_1-K_1\right)\xi}{\left(P_m+1\right)}e^{-\left(P_m+1\right)\xi}+\frac{\left(A_1-K_1\right)\left(P_m+2\right)}{\left(P_m+1\right)^2}e^{-\left(P_m+1\right)\xi}+\frac{P_mM\cos\alpha}{P_m-2}e^{-2\xi}$$

$$+\frac{2P_m\left(D_8-2D_1\right)M\cos\alpha}{P_m-1}e^{-\xi}+\frac{2P_mD_1M\cos\alpha\,\xi}{P_m-1}e^{-\xi}+\frac{2P_mD_1\left(P_m-2\right)M\cos\alpha}{\left(P_m-1\right)^2}e^{-\xi}$$

$$+\frac{2D_9P_mP_rM\cos\alpha}{P_m-P_r}e^{-P_r\xi}+\frac{2D_{10}P_mS_cM\cos\alpha}{P_m-S_c}e^{-S_c\xi}+2D_7P_m^2M\cos\alpha\,\xi e^{-P_m\xi}$$

The general solution is

$$L_2=c_1+c_2e^{-P_m\xi}+\left(A_1-K_1\right)P_me^{-(P_m+1)\xi}+\left(A_1-K_1\right)P_m\xi e^{-P_m\xi}-\frac{A_1P_m}{1-P_m}e^{-\xi}+\frac{A_1P_m\xi}{2\left(P_m-2\right)}e^{-2\xi}$$

$$+\frac{A_1\left(P_m-4\right)P_m}{4\left(P_m-2\right)^2}e^{-2\xi}+\frac{\left(A_1-K_1\right)\xi}{\left(P_m+1\right)}e^{-(P_m+1)\xi}+\frac{\left(A_1-K_1\right)\left(P_m+2\right)}{\left(P_m+1\right)^2}e^{-(P_m+1)\xi}+\frac{P_mM\cos\alpha}{P_m-2}e^{-2\xi}$$

$$+\frac{2\left(D_8-2D_1\right)P_mM\cos\alpha}{P_m-1}e^{-\xi}+\frac{2D_1P_mM\cos\alpha\,\xi}{P_m-1}e^{-\xi}+\frac{2D_1\left(P_m-2\right)P_mM\cos\alpha}{\left(P_m-1\right)^2}e^{-\xi}$$

$$+\frac{2D_9P_mP_rM\cos\alpha}{P_m-P_r}e^{-P_r\xi}+\frac{2D_{10}P_mS_cM\cos\alpha}{P_m-S_c}e^{-S_c\xi}+2D_7P_m^2M\cos\alpha\,\xi e^{-P_m\xi}$$

Using boundary conditions:

$L_2=0,$ at $\xi=0$ and $L_2=0$ as $\xi\to\infty$

Therefore, $0=c_1+c_2+\left(A_1-K_1\right)P_m+\frac{A_1P_m}{P_m-1}+\frac{A_1\left(P_m-4\right)P_m}{4\left(P_m-2\right)^2}+\frac{\left(A_1-K_1\right)\left(P_m+2\right)}{\left(P_m+1\right)^2}$

$$+\frac{P_mM\cos\alpha}{P_m-2}+\frac{2\left(D_8-2D_1\right)P_mM\cos\alpha}{P_m-1}+\frac{2D_1\left(P_m-2\right)P_mM\cos\alpha}{\left(P_m-1\right)^2}$$

$$+\frac{2D_9P_mP_rM\cos\alpha}{P_m-P_r}+\frac{2D_{10}P_mS_cM\cos\alpha}{P_m-S_c}$$

and $c_1=0$, so that

$$L_2=-\left[P_m\left(A_1-K_1\right)+\frac{A_1P_m}{P_m-1}+\frac{A_1\left(P_m-4\right)P_m}{4\left(P_m-2\right)^2}+\frac{\left(A_1-K_1\right)\left(P_m+2\right)}{\left(P_m+1\right)^2}+\frac{P_mM\cos\alpha}{P_m-2}\right.$$

$$+\frac{2\left(D_8-2D_1\right)P_mM\cos\alpha}{P_m-1}+\frac{2D_1\left(P_m-2\right)P_mM\cos\alpha}{\left(P_m-1\right)^2}+\frac{2D_9P_mP_rM\cos\alpha}{P_m-P_r}$$

$$\left.+\frac{2D_{10}P_mS_cM\cos\alpha}{P_m-S_c}\right]e^{-P_m\xi}+\left(A_1-K_1\right)P_me^{-(P_m+1)\xi}+\left(A_1-K_1\right)P_m\,\xi e^{-P_m\xi}$$

$$-\frac{A_1P_m}{1-P_m}e^{-\xi}+\frac{A_1P_m\xi}{2\left(P_m-2\right)}e^{-2\xi}+\frac{A_1\left(P_m-4\right)P_m}{4\left(P_m-2\right)^2}e^{-2\xi}+\frac{\left(A_1-K_1\right)\xi}{\left(P_m+1\right)}e^{-(P_m+1)\xi}$$

$$+\frac{\left(A_1-K_1\right)\left(P_m+2\right)}{\left(P_m+1\right)^2}e^{-(P_m+1)\xi}+\frac{P_mM\cos\alpha}{P_m-2}e^{-2\xi}+\frac{2\left(D_8-2D_1\right)P_mM\cos\alpha}{P_m-1}e^{-\xi}$$

$$+\frac{2D_1P_mM\cos\alpha\,\xi}{P_m-1}e^{-\xi}+\frac{2D_1\left(P_m-2\right)P_mM\cos\alpha}{\left(P_m-1\right)^2}e^{-\xi}+\frac{2D_9P_mP_rM\cos\alpha}{P_m-P_r}e^{-P_r\xi}$$

$$+\frac{2D_{10}P_mS_cM\cos\alpha}{P_m-S_c}e^{-S_c\xi}+2D_7P_m^2M\cos\alpha\,\xi e^{-P_m\xi}$$

Hence the complete solution of equation (3.11) is

$$L_2\left(\xi\right)=B_1e^{-(P_m+1)\xi}+B_1\xi e^{-P_m\xi}+B_2e^{-\xi}+B_3e^{-2\xi}+B_4e^{-(P_m+1)\xi}+B_5e^{-2\xi}+B_6e^{-\xi}$$

$$+B_7e^{-\xi}+B_8e^{-P_r\xi}+B_9e^{-S_c\xi}+B_{10}e^{-P_m\xi}+B_{11}\xi e^{-2\xi}$$

$$+B_{12}\xi e^{-(P_m+1)\xi}+B_{13}\xi e^{-\xi}+B_{14}\xi e^{-P_m\xi} \qquad (3.20)$$

Where $B_1=\left(A_1-K_1\right)P_m,\ B_2=\dfrac{A_1P_m}{P_m-1},\ B_3=\dfrac{A_1\left(P_m-4\right)P_m}{4\left(P_m-2\right)^2},\ B_4=\dfrac{\left(A_1-K_1\right)\left(P_m+2\right)}{\left(P_m+1\right)^2},$

$$B_5=\frac{P_mM\cos\alpha}{P_m-2},B_6=\frac{2\left(D_8-2D_1\right)P_mM\cos\alpha}{P_m-1},\ B_7=\frac{2D_1\left(P_m-2\right)P_mM\cos\alpha}{\left(P_m-1\right)^2},$$

$$B_8=\frac{2D_9P_mP_rM\cos\alpha}{P_m-P_r},\ B_9=\frac{2D_{10}P_mS_cM\cos\alpha}{P_m-S_c},$$

$$B_{10}=-\left[B_1+B_2+B_3+B_4+B_5+B_6+B_7+B_8+B_9\right],B_{11}=\frac{A_1P_m}{2\left(P_m-2\right)},$$

$$B_{12}=\frac{A_1-K_1}{P_m+1},\ B_{13}=\frac{2D_1P_mM\cos\alpha}{P_m-1},\ B_{14}=2D_7P_m^2M\cos\alpha$$

Again from equation (3.12) we have

$$G_2''+P_r\left(G_2'+F_1G_1'\right)+\frac{E_cP_r}{2P_m}\left(L_1'\right)^2=0$$

Here $G_1=e^{-P_r\xi},G_1'=-P_re^{-P_r\xi}$

$$F_1=1-e^{-\xi},\ L_1=A_1\left(e^{-\xi}-e^{-P_m\xi}\right)+K_1e^{-P_m\xi},L_1'=A_1\left(P_me^{-P_m\xi}-e^{-\xi}\right)-K_1P_me^{P_m\xi}$$

$$\left(L_1'\right)^2=\left(A_1-K_1^2\right)P_m^2e^{-2P_m\xi}+\left(2A_1K_1P_m-2A_1^2P_m\right)e^{-(P_m+1)\xi}-A_1^2e^{-2\xi}$$

The complementary function is obtained by

$$G_{2c}=c_1+c_2e^{-P_r\xi}$$

and the particular integral is

$$G_{2p}=-P_r\xi e^{-P_r\xi}-\frac{P_r^2}{\left(P_r+1\right)}e^{-(P_r+1)\xi}-\frac{P_rE_c\left(A_1-K_1\right)^2}{4\left(2P_m-P_r\right)}e^{-2P_m\xi}+\frac{A_1P_rE_c\left(A_1-K_1\right)}{\left(P_m+1\right)\left(P_m-P_r+1\right)}e^{-(P_m+1)\xi}$$

$$-\frac{P_rE_cA_1^2}{4P_m\left(2-P_r\right)}e^{-2\xi}$$

The general solution is

$$G_2 = c_1 + c_2 e^{-P_r\xi} - P_r\xi e^{-P_r\xi} - \frac{P_r^2}{(P_r+1)} e^{-(P_r+1)\xi} - \frac{(A_1-K_1)^2 P_r E_c}{4(2P_m - P_r)} e^{-P_m\xi}$$
$$+\frac{A_1(A_1-K_1)P_rE_c}{(P_m+1)(P_m-P_r+1)} e^{-(P_m+1)\xi} - \frac{A_1^2 P_r E_c}{4P_m(2-P_r)} e^{-2\xi}$$

Using boundary conditions:

$G_2 = 0$ at $\xi = 0$ and $G_2 = 0$ as $\xi \to \infty$

Therefore, $0 = c_1 + c_2 - \dfrac{P_r^2}{(P_r+1)} - \dfrac{(A_1-K_1)^2 P_rE_c}{4(2P_m-P_r)} + \dfrac{A_1(A_1-K_1)P_rE_c}{(P_m+1)(P_m-P_r+1)} - \dfrac{A_1^2P_rE_c}{4P_m(2-P_r)}$

and $c_1 = 0$, so that

$$G_2(\xi) = A_4\xi e^{-P_r\xi} - A_5 e^{-(P_r+1)\xi} - A_6 e^{-2P_m\xi} + A_7 e^{-(P_m+1)\xi} - A_8 e^{-2\xi} + A_9 e^{-P_r\xi} \tag{3.21}$$

where $A_4 = -P_r$, $A_5 = \dfrac{P_r^2}{P_r+1}$, $A_6 = \dfrac{(A_1-K_1)^2 P_rE_c}{4(2P_m-P_r)}$, $A_7 = \dfrac{A_1(A_1-K_1)P_rE_c}{(P_m+1)(P_m-P_r+1)}$,

$A_8 = \dfrac{A_1^2P_rE_c}{4P_m(2-P_r)}$, $A_9 = A_5 + A_6 - A_7 + A_8$

Again from equation (3.13) we have

$P_2'' - 2S_cP_1F_1' + S_c(P_1'F_1 + P_2') + S_0S_cG_2'' = 0$

Here

$F_1 = 1 - e^{-\xi}$, $F_1' = e^{-\xi}$, $P_1 = A_2e^{-S_c\xi} + A_3e^{-P_c\xi}$

$P_1' = -A_2S_ce^{-S_c\xi} - A_3P_re^{-P_r\xi}$

$G_2 = A_4\xi e^{-P_r\xi} - A_5e^{-(P_r+1)\xi} - A_6e^{-2P_m\xi} + A_7e^{-(P_m+1)\xi} - A_8e^{-2\xi} + A_9e^{-P_r\xi}$

$G_2' = A_4e^{-P_r\xi} - A_4P_r\xi e^{-P_r\xi} + A_5(P_r+1)e^{-(P_r+1)\xi} + 2A_6P_me^{-2P_m\xi}$

$\quad -A_7(P_m+1)e^{-(P_m+1)\xi} + 2A_8e^{-2\xi} - A_9P_re^{-P_r\xi}$

$G_2'' = -A_4P_re^{-P_r\xi} - A_4P_re^{-P_r\xi} + A_4\xi P_r^2e^{-P_r\xi} - A_5(P_r+1)^2e^{-(P_r+1)\xi}$

$\quad -4A_6P_m^2e^{-2P_m\xi} + A_7(P_m+1)^2e^{-(P_m+1)\xi} - 4A_8e^{-2\xi} + A_9P_r^2e^{-P_r\xi}$

The complementary function is obtained by

$P_{2c} = c_1 + c_2e^{-S_c\xi}$

and the particular integral

$$P_{2p} = \frac{2A_2S_c - A_2S_c^2}{(S_c+1)}e^{-(S_c+1)\xi} + \frac{2A_3S_c - A_3P_rS_c + A_5S_0S_c(P_r+1)^2}{(P_r+1)(P_r-S_c+1)}e^{-(P_r+1)\xi} - A_2S_c\xi e^{-S_c\xi}$$

$$+\frac{A_3S_c+2A_4S_0S_c-A_9S_0S_cP_r}{P_r-S_c}e^{-P_r\xi}-\frac{A_4S_cS_0P_r}{P_r-S_c}\xi e^{-P_r\xi}+\frac{A_4S_cS_0}{\left(P_r-S_c\right)^2}\left(S_c-2P_r\right)e^{-P_r\xi}$$

$$+\frac{2A_6S_cS_0P_m}{2P_m-S_c}e^{-2P_m\xi}-\frac{A_7S_oS_c\left(P_m+1\right)}{\left(P_m-S_c+1\right)}e^{-(P_m+1)\xi}+\frac{2A_8S_oS_c}{2-S_c}e^{-2\xi}$$

The general solution is

$$P_2=c_1+c_2e^{-S_c\xi}+\frac{2A_2S_c-A_2S_c^2}{\left(S_c+1\right)}e^{-(S_c+1)\xi}+\frac{2A_3S_c-A_3P_rS_c+A_5S_oS_c\left(P_r+1\right)^2}{\left(P_r+1\right)\left(P_r-S_c+1\right)}e^{-(P_r+1)\xi}$$

$$-A_2S_c\xi e^{-S_c\xi}+\frac{A_3S_c+2A_4S_0S_c-A_9S_0S_cP_r}{P_r-S_c}e^{-P_r\xi}-\frac{A_4S_cS_0P_r}{P_r-S_c}\xi e^{-P_r\xi}$$

$$+\frac{A_4S_cS_0}{\left(P_r-S_c\right)^2}\left(S_c-2P_r\right)e^{-P_r\xi}+\frac{2A_6S_cS_0P_m}{2P_m-S_c}e^{-2P_m\xi}$$

$$-\frac{A_7S_cS_0\left(P_m+1\right)}{\left(P_m-S_c+1\right)}e^{-(P_m+1)\xi}+\frac{2A_8S_cS_c}{2-S_c}e^{-2\xi}$$

Using boundary conditions:

$P_2=0$ at $\xi=0$ and $G_2=0$ as $\xi\to\infty$

Therefore, $0=c_1+c_2+\frac{2A_2S_c-A_2S_c^2}{\left(S_c+1\right)}+\frac{2A_3S_c-A_3P_rS_c+A_5S_0S_c\left(P_r+1\right)^2}{\left(P_r+1\right)\left(P_r-S_c+1\right)}$

$$+\frac{A_3S_c+2A_4S_0S_c-A_9S_oS_cP_r}{P_r-S_c}+\frac{A_4S_cS_0}{\left(P_r-S_c\right)^2}\left(S_c-2P_r\right)+\frac{2A_6S_cS_0P_m}{2P_m-S_c}-\frac{A_7S_oS_c\left(P_m+1\right)}{\left(P_m-S_c+1\right)}+\frac{2A_8S_0S_c}{2-S_c}$$

and $c_1=0$, so that

$$P_2\left(\xi\right)=E_1e^{-P_r\xi}+E_2e^{-(P_r+1)\xi}+E_3e^{-(S_c+1)\xi}+E_4e^{-2P_m\xi}-E_5e^{-(P_m+1)\xi}$$
$$+E_6e^{-2\xi}+E_7e^{-P_r\xi}+E_8e^{-S_c\xi}-E_9\xi e^{-S_c\xi}-E_{10}\xi e^{-P_r\xi} \tag{3.22}$$

The solution of equations (3.5)–(3.8) and (3.10)–(3.13) up to the second order under the prescribed boundary conditions (3.9) and (3.14), respectively are obtained in a straightforward manner and are re-written as:

$$F_1\left(\xi\right)=1-e^{-\xi} \tag{3.23}$$

$$L_1\left(\xi\right)=A_1\left(e^{-\xi}-e^{-P_m\xi}\right)+K_1e^{-P_m\xi} \tag{3.24}$$

$$G_1\left(\xi\right)=e^{-P_r\xi} \tag{3.25}$$

$$P_1\left(\xi\right)=A_2e^{-S_c\xi}+A_3e^{-P_r\xi} \tag{3.26}$$

$$F_2\left(\xi\right)=\frac{1}{4}e^{-2\xi}+D_1\xi e^{-\xi}+D_9e^{-P_r\xi}+D_{10}e^{-S_c\xi}+D_7e^{-P_m\xi}+D_8e^{-\xi}+D_{11} \tag{3.27}$$

$$L_2(\xi)=B_1e^{-(P_m+1)\xi}+B_1\xi e^{-P_m\xi}+B_2e^{-\xi}+B_3e^{-2\xi}+B_4e^{-(P_m+1)\xi}+B_5e^{-2\xi}+B_6e^{-\xi}+B_7e^{-\xi}+B_8e^{-P_r\xi}$$
$$+B_9e^{-S_c\xi}+B_{10}e^{-P_m\xi}+B_{11}\xi e^{-2\xi}+B_{12}\xi e^{-(P_m+1)\xi}+B_{13}\xi e^{-\xi}+B_{14}\xi e^{-P_m\xi} \quad (3.28)$$

$$G_2(\xi)=A_4\xi e^{-P_r\xi}-A_5e^{-(P_r+1)\xi}-A_6e^{-2P_m\xi}+A_7e^{-(P_m+1)\xi}-A_8e^{-2\xi}+A_9e^{-P_r\xi} \quad (3.29)$$

$$P_2(\xi)=E_1e^{-P_r\xi}+E_2e^{-(P_r+1)\xi}+E_3e^{-(S_c+1)\xi}+E_4e^{-2P_m\xi}+E_5e^{-(P_m+1)\xi}+E_6e^{-2\xi}+E_7e^{-P_r\xi}$$
$$+E_8e^{-S_c\xi}-E_9\xi e^{-S_c\xi}-E_{10}\xi e^{-P_r\xi} \quad (3.30)$$

where the constants A_i, B_i, D_i, E_i and K_1 are shown in Appendix 3.A.

The above solutions (3.23)–(3.30) are however valid for $P_r = S_c \neq 1$ and $P_r \neq S_c$

The velocity, temperature, induced magnetic field and the mass concentration can now be calculated from (3.1)–(3.4) as follows:

$$\frac{u}{U_0}=f'(\eta)=F_1'+\varepsilon F_2'+\varepsilon^2F_3' \quad (3.31)$$

$$H(\eta)=\sqrt{\varepsilon}L_1+\varepsilon^{3/2}L_2 \quad (3.32)$$

$$\theta(\eta)=G_1+\varepsilon G_2+\varepsilon^2G_3 \quad (3.33)$$

$$\varphi(\eta)=P_1+\varepsilon P_2+\varepsilon^2P_3 \quad (3.34)$$

Thus with the help of the solutions (3.23) – (3.30) the velocity, temperature, inclined magnetic field and concentration distributions are calculated from (3.31) – (3.34). However for different values of the established parameters and numbers, the results of the velocity, temperature, inclined magnetic field and concentration distribution are plotted graphically and the coefficients of skin friction, wall heat flux and are wall mass flux are presented in tabular form in CHAPTER IV.

Appendix 3.A

$$A_1 = \frac{2P_m M\cos\alpha}{1-P_m} \qquad K_1 = \frac{1}{\sqrt{\varepsilon}} \qquad A_2 = 1 - A_3, \qquad A_3 = -\frac{S_0 S_c P_r}{P_r - S_c}$$

$$D_1 = 1 + A_1 M\cos\alpha, \qquad D_2 = G_r + G_m A_3, \qquad D_3 = G_m A_2, \qquad D_4 = \frac{D_2}{P_r\left(P_r - 1\right)},$$

$$D_5 = \frac{D_3}{S_c\left(S_c - 1\right)}, \qquad D_6 = \frac{D_{12}}{P_m - 1}, \qquad D_7 = \frac{D_{12}}{P_m\left(P_m - 1\right)},$$

$$D_8 = -\frac{1}{2} + D_1 - D_4 - D_5 - D_6, \qquad D_9 = \frac{D_2}{P_r^2\left(P_r - 1\right)}, \qquad D_{10} = \frac{D_3}{S_c^2\left(S_c - 1\right)},$$

$$D_{11} = -\frac{1}{4} - D_8 - D_9 - D_{10} - D_7, \qquad D_{12} = \left(A_1 - K_1\right)M\cos\alpha,$$

$$B_1 = P_m\left(A_1 - K_1\right), \qquad B_2 = \frac{A_1 P_m}{P_m - 1}, \qquad B_3 = \frac{A_1\left(P_m - 4\right)P_m}{4\left(P_m - 2\right)^2} \qquad B_4 = \frac{\left(A_1 - K_1\right)\left(P_m + 2\right)}{\left(P_m + 1\right)^2}$$

$$B_5 = \frac{P_m M\cos\alpha}{P_m - 2}, \qquad B_6 = \frac{2\left(D_8 - 2D_1\right)P_m M\cos\alpha}{P_m - 1} \qquad B_7 = \frac{2D_1\left(P_m - 2\right)P_m M\cos\alpha}{\left(P_m - 1\right)^2},$$

$$B_8 = \frac{2D_9 P_m P_r M\cos\alpha}{P_m - P_r} \qquad B_9 = \frac{2D_{10} P_m S_c M\cos\alpha}{P_m - S_c},$$

$$B_{10} = -\left[B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9\right], \qquad B_{11} = \frac{A_1 P_m}{2\left(P_m - 2\right)}, \; B_{12} = \frac{A_1 - K_1}{P_m + 1}$$

$$B_{13} = \frac{2D_1 P_m M\cos\alpha}{P_m - 1}, \qquad B_{14} = 2D_7 P_m^2 M\cos\alpha \qquad A_4 = -P_r \qquad A_5 = \frac{P_r^2}{P_r + 1}$$

$$A_6 = \frac{\left(A_1 - K_1\right)^2 P_r E_c}{4\left(2P_m - P_r\right)}, \qquad A_7 = \frac{A_1\left(A_1 - K_1\right)P_r E_c}{\left(P_m + 1\right)\left(P_m - P_r + 1\right)}, \; A_8 = \frac{A_1^2 P_r E_c}{4P_m\left(2 - P_r\right)}, \; A_9 = A_5 + A_6 - A_7 + A_8$$

$$E_1 = \frac{A_3 S_c + 2A_4 S_0 S_c - A_9 P_r S_o S_c}{P_r - S_c}, \qquad E_2 = \frac{2A_3 S_c - A_3 P_r S_c + A_5 S_o S_c\left(P_r + 1\right)^2}{\left(P_r + 1\right)\left(P_r - S_c + 1\right)},$$

$$E_3 = \frac{2A_2 S_c - A_2 S_c^2}{\left(S_c + 1\right)}, \qquad E_4 = \frac{2A_6 S_c S_o P_m}{2P_m - S_c} \qquad E_5 = \frac{A_7 S_o S_c\left(P_m + 1\right)}{\left(P_m - S_c + 1\right)}$$

$$E_6 = \frac{2A_8 S_0 S_c}{2 - S_c} \qquad E_7 = \frac{A_4 S_c S_o}{\left(P_r - S_c\right)^2}\left(S_c - 2P_r\right), \qquad E_6 = \frac{2A_8 S_0 S_c}{2 - S_c}$$

$$E_7 = \frac{A_4 S_c S_0}{\left(P_r - S_c\right)}\left(S_c - 2P_r\right), \qquad E_8 = -E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7,$$

$$E_9 = A_2 S_c \qquad E_{10} = \frac{A_4 P_r S_c S_0}{P_r - S_c}$$

CHAPTER IV

Perturbation Solutions and Results Discussions

4.1 Numerical Solution

The solution of system of coupled, nonlinear, ordinary differential equations (3.5)–(3.8) and (3.10)–(3.13) together with the boundary conditions (3.9) and (3.14) respectively, are obtained by using perturbation technique. In order to put the physical insight into the flow pattern of the problem, the approximate numerical results of the first order solutions (3.23)–(3.26) along with the second order solutions (3.27)–(3.30), concerning the velocity, temperature, induced magnetic field and concentration field are obtained. For more consistent results the numerical approximation of the second order solutions (3.27)–(3.30), in collaboration with the first order approximation, have been carried out here for small values of Eckert number $E_c = 0.2$ (which is the measure of the heat produced by friction) with different selected values of the established dimensionless numbers and parameters like Soret number (S_0), Grashof number (G_r), Modified Grashof number (G_m) for mass transfer, suction parameter f_w, magnetic parameter (M), etc. Since the two most important fluids are atmospheric air and water, the values of the Prandtl number (P_r) are chosen to be 0.71 for air (at $20°C$) to 7.0 for water (at $20°C$) for numerical investigation. The others like magnetic diffusivity parameter (P_m) and Schmidt number (S_c) are chosen to be the fixed values 3.0, 0.6, respectively.

As one of the main intentions of this research is to investigate the influence of inclined magnetic field on the above mentioned field variables, three different inclined angles of uniform applied magnetic field have been considered, namely, 0^0, 45^0 and 75^0 for the present problem. However, to show the effects of non-dimensional numbers or parameters on the flow fields, namely, velocity, temperature, concentration and induced magnetic fields, the inclined angle is kept 45^0everywhere. Besides, to show the variation effects of any one of the non-dimensional numbers or parameters on the flow fields, the other numbers/parameters have been taken to be fixed values. With the above considerations the velocity and temperature profiles, the induced magnetic field and the mass concentration are obtained for some selected values of the inclined angles, non-dimensional numbers and parameters. The dependency of wall shear stress, wall heat flux, and wall mass flux, which are of physical interests, on the concerned non-dimensional parameters and numbers are also been calculated.

4.2 Numerical Results and Discussions

In the following, the obtained numerical results involving the effects of the variation of considered inclination angles, non-dimensional numbers and parameters on the velocity profile, temperature profile, induced magnetic field and concentration distribution have been presented through Figures 4.2.1– 4.2.24 whereas the variation of values proportional to the skin friction coefficient $\left(f''(0)\right)$, Nusselt number $\left(-\theta'(0)\right)$ and Sherwood number $\left(-\varphi'(0)\right)$ with the variation of the values of different selected established dimensionless parameters and numbers are tabulated and have been illustrated through Tables 4.2.1– 4.2.7.

The effects of the variations of the considered inclined angle on the velocity, temperature, concentration and induced magnetic fields have been presented through Figures 4.2.1 – 4.2.4. Figure 4.2.1 represents the velocity profiles for the variation of the angle of inclination of applied magnetic field. It is observed that in all cases the velocity starts from 1.0 and with the increase of the distance from the plate the velocities increased rapidly and after certain distance the velocity decreases and then leads to zero asymptotically which clearly shows the boundary layer effect. Also profiles show that velocity decrease with the increase of the inclined angle α from 0^0 to 75^0. This Implies that the angle of inclination has a retarding influence on the velocity.

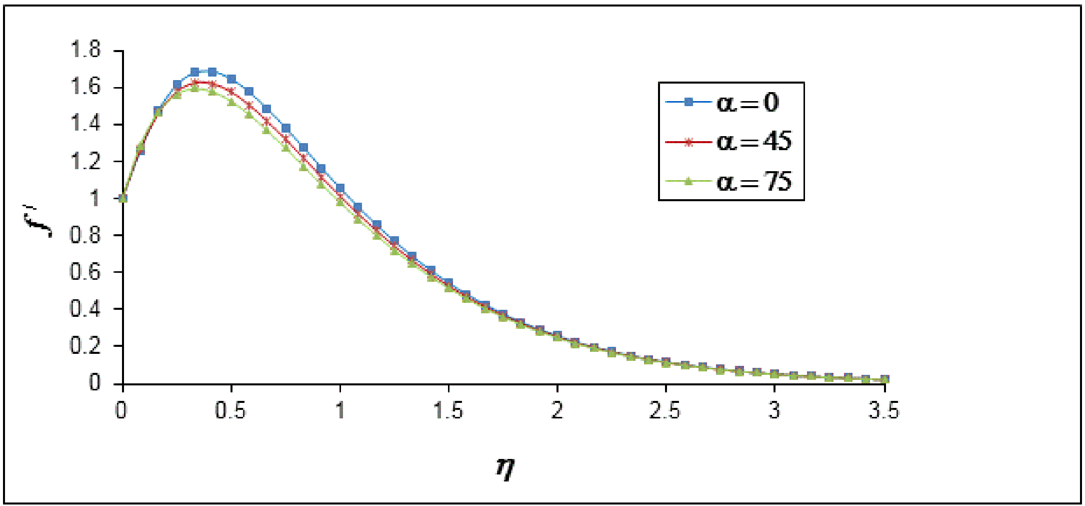

Figure 4.2.1: Effect inclined angle α (in degree) on velocity fields (with fixed values of $P_r = 0.71, S_0 = 2.0, S_c = 0.6, M = 1.5, G_r = 6.0, f_w = 3.0$ and $G_m = 4.0$).

Figure 4.2.2 shows the temperature profiles for different inclined angle. In fact the variable considered is not the temperature itself, rather it is the ratio of the difference between temperature of the fluid within the boundary layer and that of the fluid outside the boundary layer to the difference between temperature at the wall and temperature outside of the boundary layer. It is seen that the temperature starts at 1.0 and with the increase of the distance from the plate the temperature decreases rapidly within the boundary layer. In all cases the temperature reduces to zero. But the temperature profiles don't affect by any means with the variation of the inclined angle as shown in Figure 4.2.2.

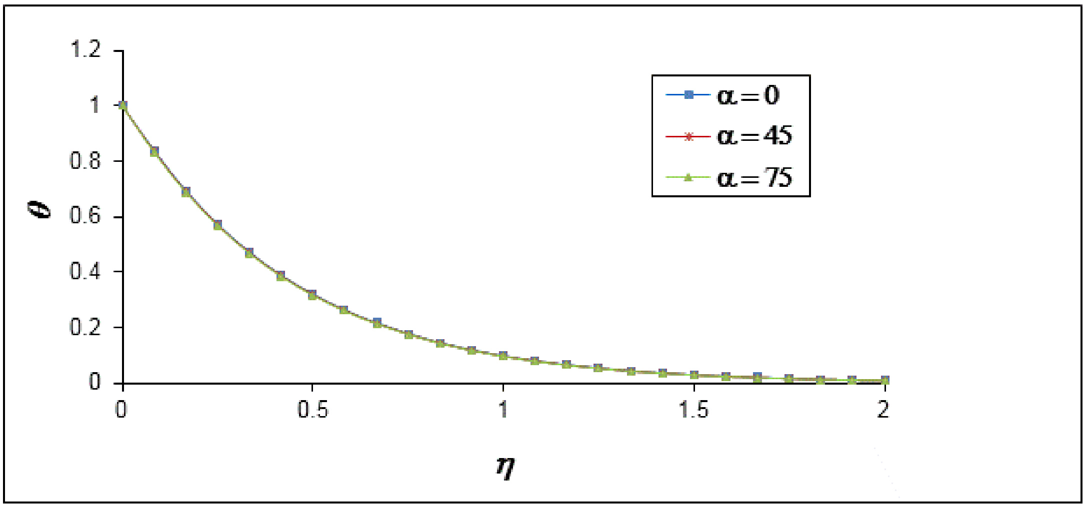

Figure 4.2.2: Effect inclined angle α (in degree) on temperature fields (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $G_r = 6.0$, $f_w = 3.0$ and $G_m = 4.0$).

The effect of the variations of inclined angle on the induced magnetic field is very strong as seen in Figure 4.2.3. From figure it is seen that with the increase in inclined angle α, the magnitude of the induced magnetic field decreases highly. The concentration fields affect a little only inside the boundary layer and are decreased with the increase of α was observed in Figure 4.2.4.

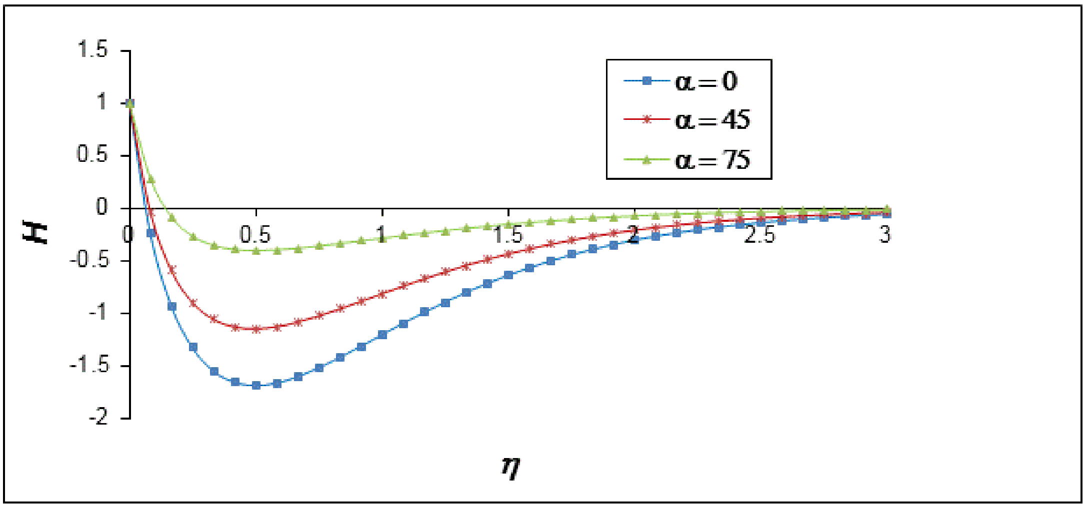

Figure 4.2.3: Effect inclined angle α (in degree) on induced magnetic fields (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $G_r = 6.0$, $f_w = 3.0$ and $G_m = 4.0$).

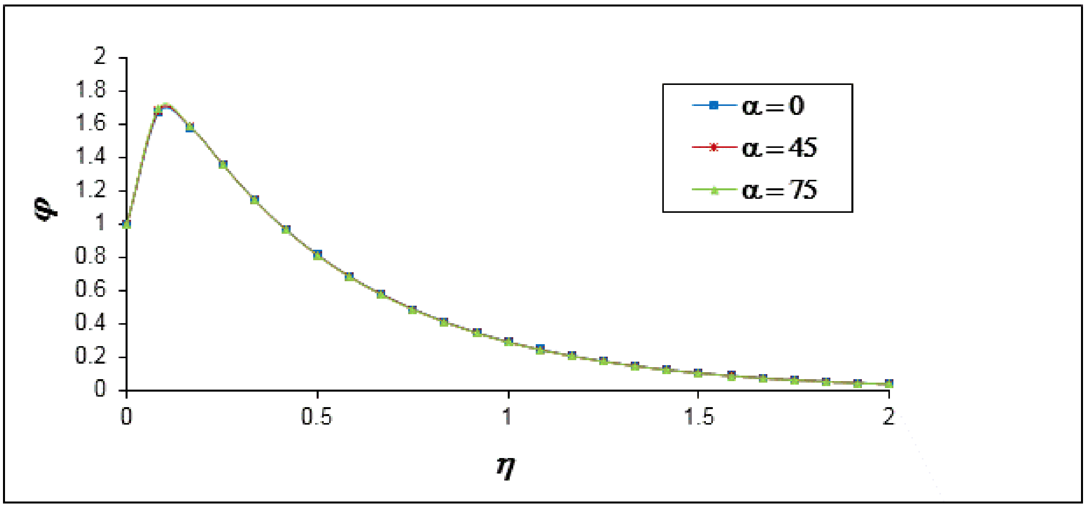

Figure 4.2.4: Effect inclined angle α (in degree) on concentration fields (with fixed values of $P_r = 0.71,\ S_0 = 2.0,\ S_c = 0.6,\ M = 1.5,\ G_r = 6.0,\ f_w = 3.0$ and $G_m = 4.0$).

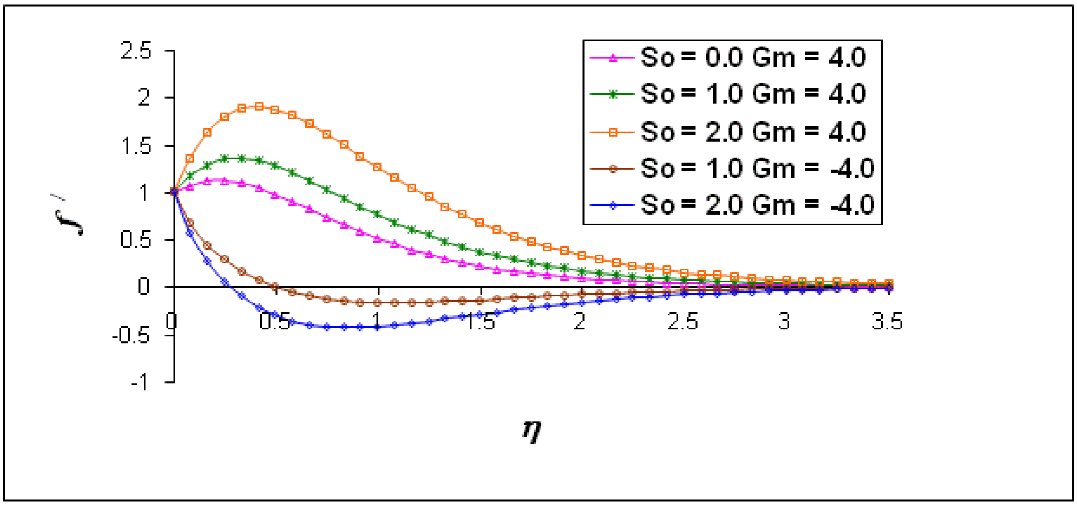

Figure 4.2.5: Velocity profiles for different values of S_0 (with fixed values of $P_r = 0.71$, $S_c = 0.6,\ M = 1.5,\ G_r = 6.0,\ f_w = 3.0,\ \alpha = 45^0$) taking $G_m = 4.0$ and -4.0.

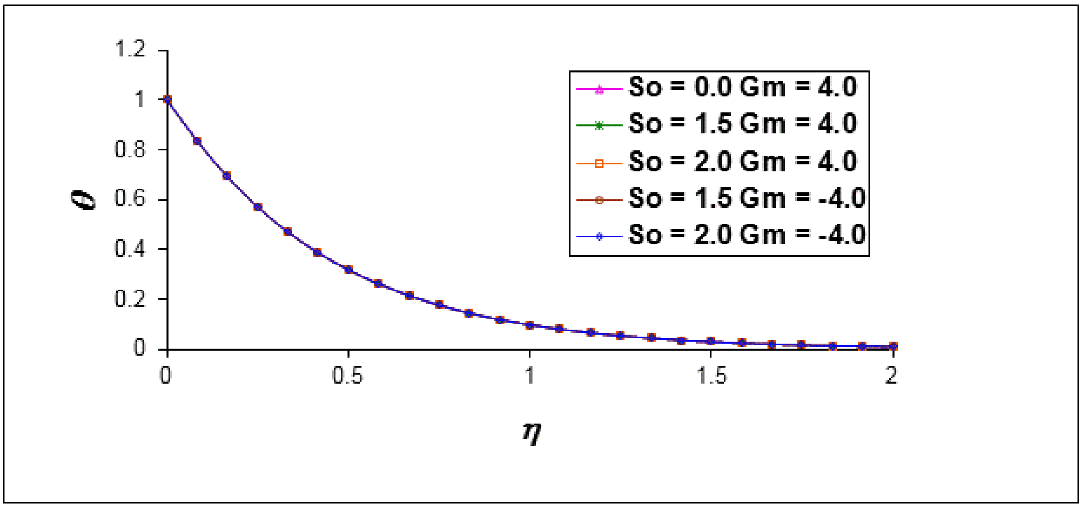

Figure 4.2.6: Temperature profiles for different values of S_0 (with fixed values of $P_r = 0.71$. $S_c = 0.6,\ M = 1.5,\ G_r = 6.0,\ f_w = 3.0,\ \alpha = 45^0$) taking $G_m = 4.0$ and -4.0.

Figures 4.2.5 and 4.2.6 show the effect of Soret number (S_0) on the velocity and temperature fields respectively. It is observed that velocity increases with the increase of S_0 but there is no remarkable effect of S_0 on the temperature field. Also the velocity decreases more with increasing S_0 for negative values of modified Grashof number G_m and it becomes negative and asymptotically tends zero far away from the plate surface.

The variation of induced magnetic field and mass concentration with S_0 are shown in Figure 4.2.7 and Figure 4.2.8 respectively. It is seen from Figure 4.2.7 that with the increase in S_0, the magnitude of the induced magnetic field increases. The magnitude of the induced magnetic field is much smaller for negative values of G_m and show decreasing with increasing S_0 as observed in Figure 4.2.7.

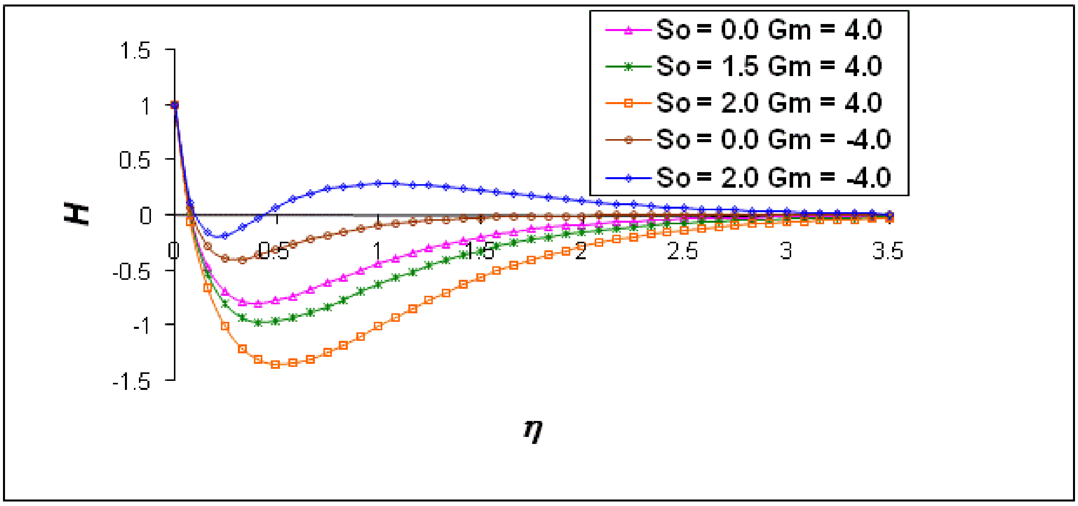

Figure 4.2.7: Induced magnetic field distribution for different values of S_0 (with fixed values of $P_r = 0.71, S_c = 0.6, M = 1.5, G_r = 6.0, f_w = 3.0, \alpha = 45^0$) taking $G_m = 4.0$ and -4.0.

The reverse effect is observed for the mass concentration, that is, concentration increases with increase of Soret number S_0 as is seen in Figure 4.2.8.

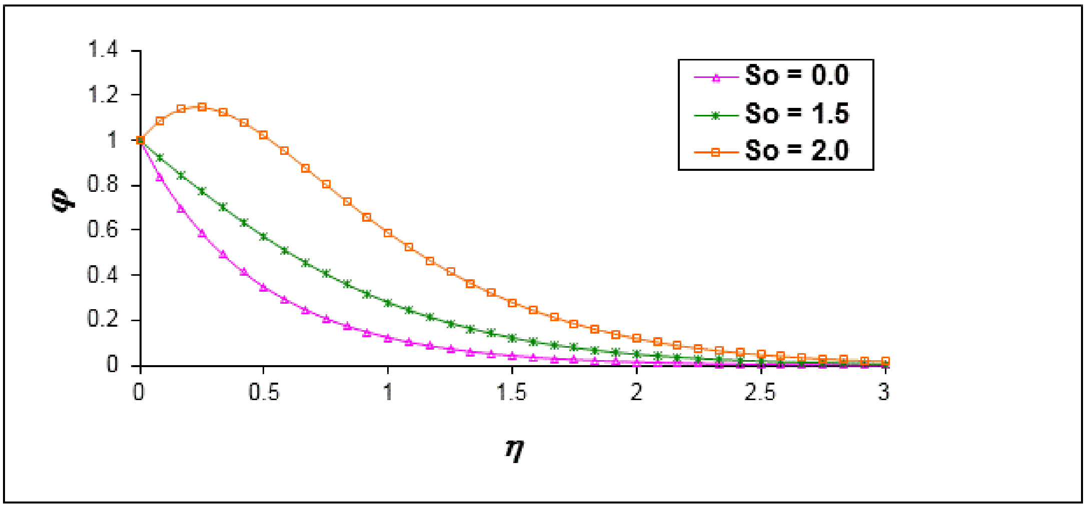

Figure 4.2.8: Concentration distribution for different values of S_0 (with fixed values of $P_r = 0.71$, $S_c = 0.6$, $M = 1.5$, $G_r = 6.0$, $f_w = 3.0$ and $G_m = 4.0$, $\alpha = 45^0$).

Figure 4.2.9 represents the velocity profiles for the variation of the suction parameter (f_w), keeping the other variables/parameters fixed. It has been observed that for increasing values of suction parameter, the velocity at any point within the boundary layer decreases rapidly. In fact the velocity within the boundary layer becomes much more slower with the increase of f_w which reflects the stability effect of the boundary layer clearly. The effects of suction parameter (f_w) on the temperature fields are presented in Figure 4.2.10. Like before, temperature found decreasing significantly with increasing suction.

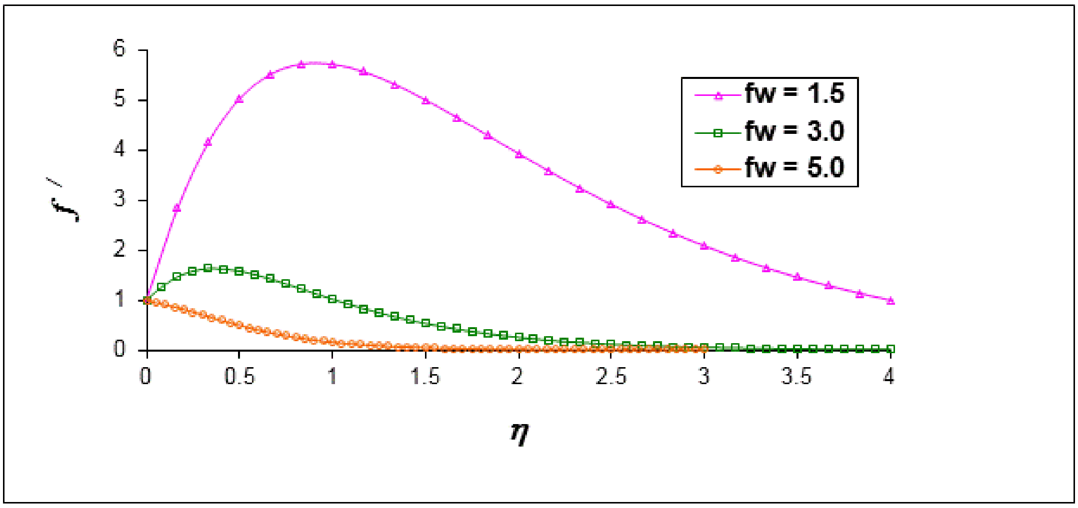

Figure 4.2.9: Velocity profiles for different values of f_w (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

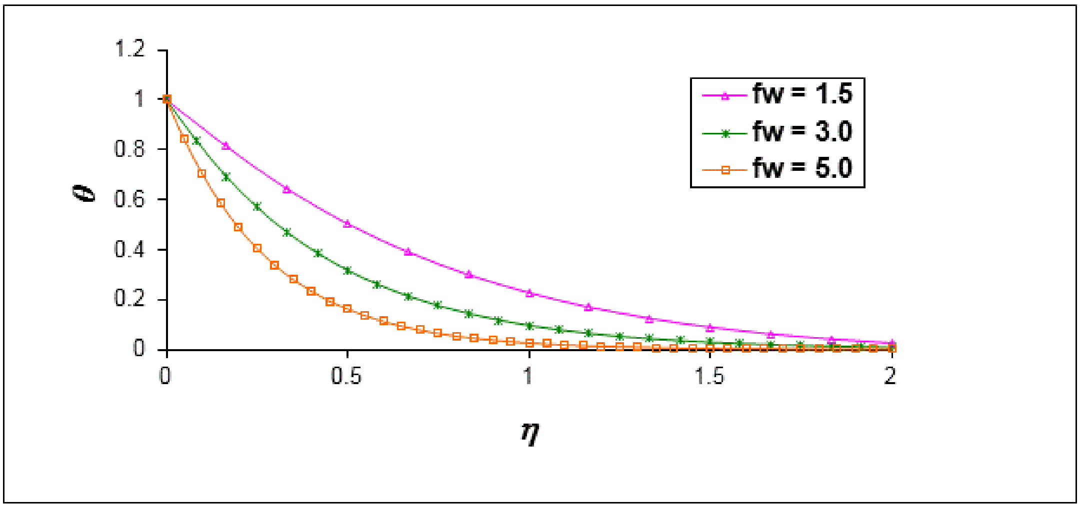

Figure 4.2.10: Temperature profiles for different values of f_w (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

Figure 4.2.11 and 4.2.12 respectively show the effect of f_w on the induced magnetic field and mass concentration. The magnitude of the induced magnetic field surprisingly decreases with the increase of f_w but concentration first increases very close to the plate surface and then found to decrease further with increasing f_w away from the surface.

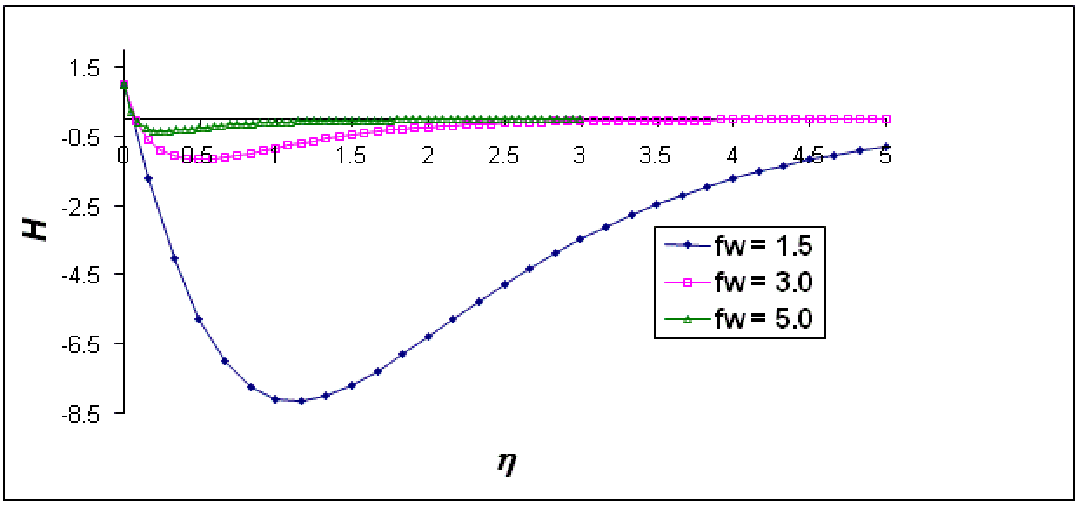

Figure 4.2.11: Induced magnetic field distribution for different values of f_w (with fixed values of $P_r = 0.71,\ S_0 = 2.0, S_c = 0.6, M = 1.5, G_r = 6.0,\ G_m = 4.0$ and $\alpha = 45^0$).

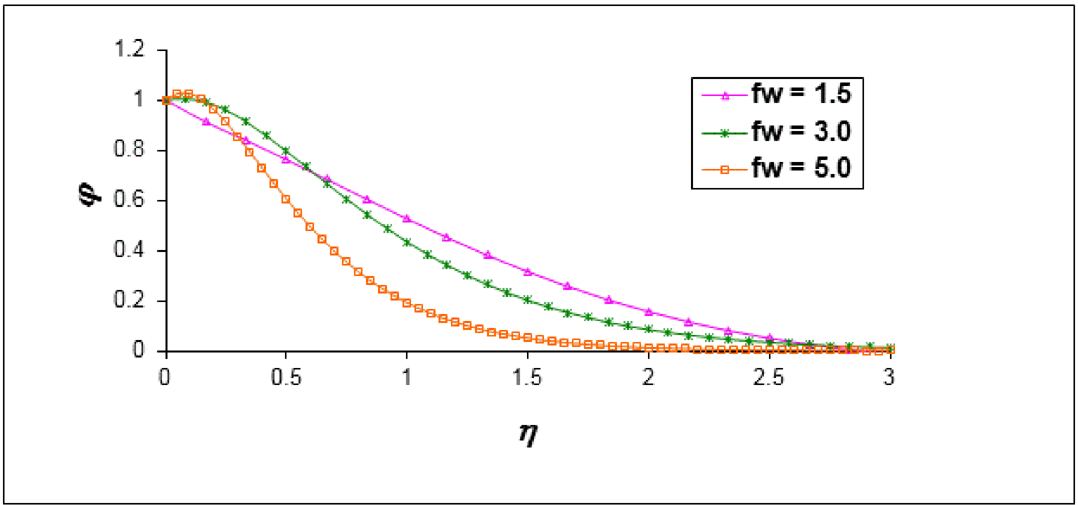

Figure 4.2.12: Concentration distribution for different values of f_w (with fixed values of $P_r = 0.71,\ S_0 = 2.0, S_c = 0.6, M = 1.5, G_r = 6.0,\ G_m = 4.0$ and $\alpha = 45^0$).

The effect of Grashof number (G_r) on the velocity and temperature fields, induced magnetic field and mass concentration are displayed in Figures 4.2.13, 4.2.14, 4.2.15 and 4.2.16, respectively. Figure 4.2.13 shows that as the values of Grashof number G_r increases, the velocity increases. From Figure 4.2.15 it is observed that the magnitude of induced magnetic fields increase with the increasing values of G_r. No considerable effect of G_r on the temperature and concentration fields is found as seen in Figure 4.2.14 and Figure 4.2.16, respectively.

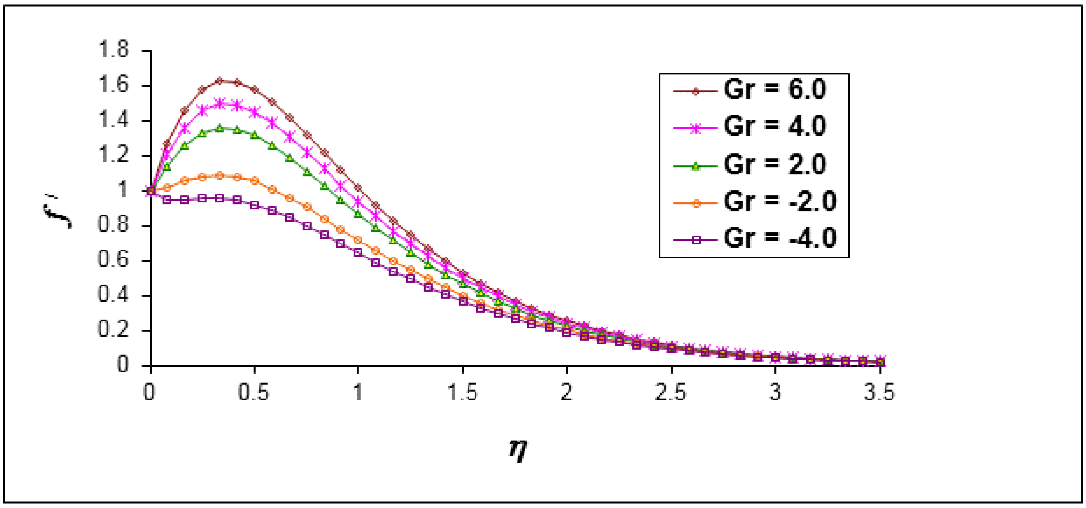

Figure 4.2.13: Velocity profiles for different values of G_r (with fixed values of $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $f_w = 3.0$, $G_m = 4.0$ and $\alpha = 45^0$).

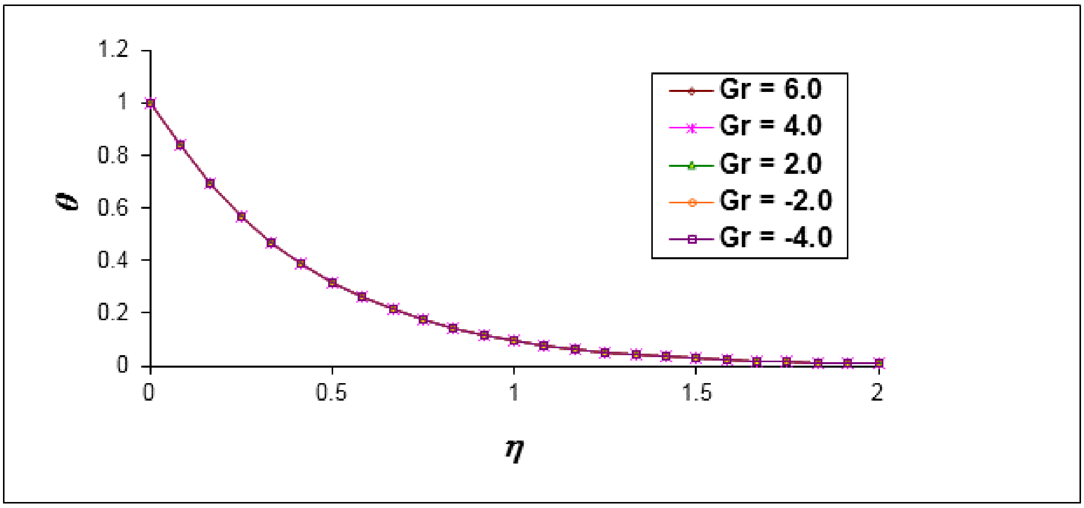

Figure 4.2.14: Temperature profiles for different values of G_r (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $f_w = 3.0$, $G_m = 4.0$ and $\alpha = 45^0$).

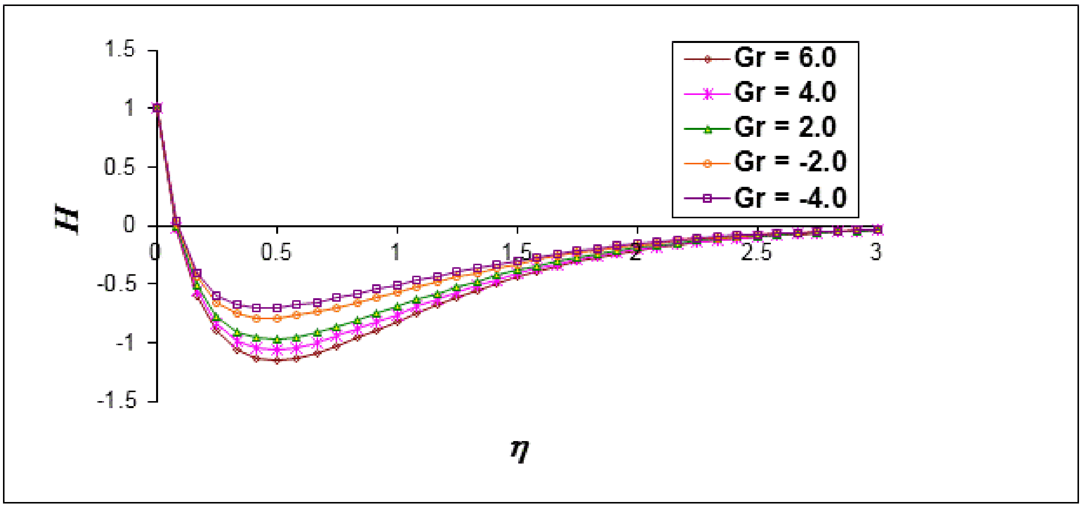

Figure 4.2.15: Induced magnetic field distribution for different values of G_r (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $f_w = 3.0$, $G_m = 4.0$ and $\alpha = 45^0$).

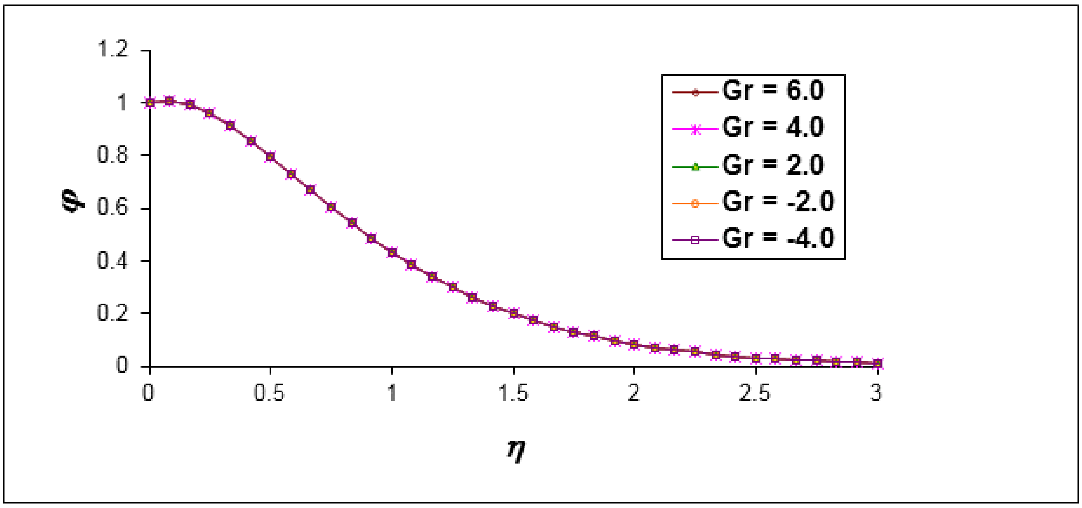

Figure 4.2.16: Concentration distribution for different values of G_r (with fixed values of $P_r = 0.71,\ S_0 = 2.0, S_c = 0.6,\ M = 1.5,\ f_w = 3.0$, $G_m = 4.0$ and $\alpha = 45^0$).

Figure 4.2.17 and Figure 4.2.18 observed the effect of magnetic parameter (M) on the velocity and temperature fields. The effect of magnetic parameter (M) on induced magnetic field and mass concentration are shown in Figures 4.2.19 and 4.2.20, respectively. As the values of M increase, both the velocity and the magnitude of induced magnetic field increase as are shown in Figures 4.2.17 and 4.2.19, respectively. From Figure 4.2.17 it is also depict that with the increase of the magnetic parameter M, the velocity profiles increase with very small difference between $0 \le \eta \le 0.25$ (approx.). After that they coincide for large value of η and finally approach to zero. Figure 4.2.19 illustrates the influence of magnetic parameter M on the induced magnetic field profiles. It is observed that as the values of M increase, the magnitude of induced magnetic fields boundary layer thickness increase. This is due to the fact that the applied magnetic field exerts additional Lorentz forces which lead to increase the induced magnetic field. But no significant effect is observed on the temperature and concentration profiles for the variation of magnetic parameter M as are observed in Figure 4.2.18 and Figure 4.2.20, respectively.

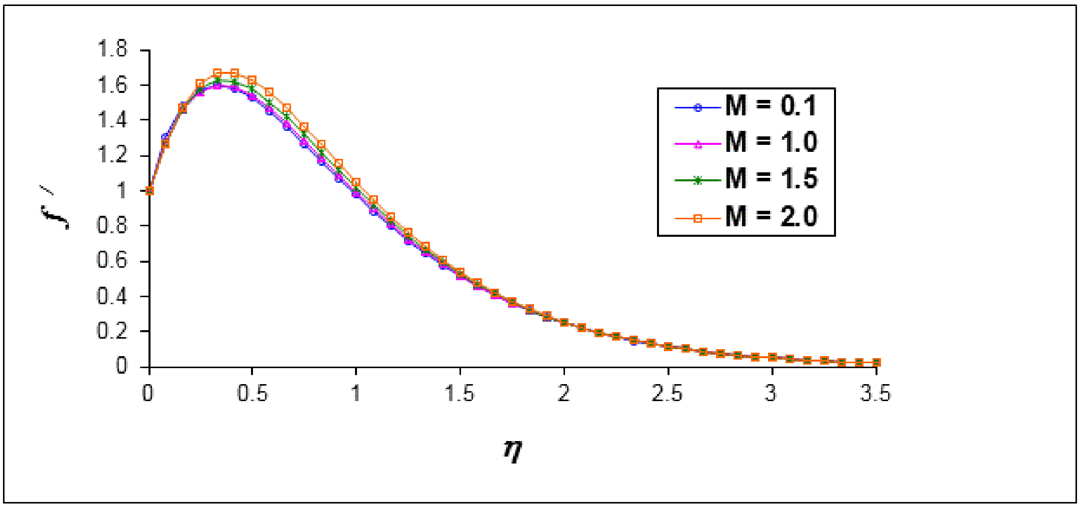

Figure 4.2.17: Velocity profiles for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $f_w = 3.0$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

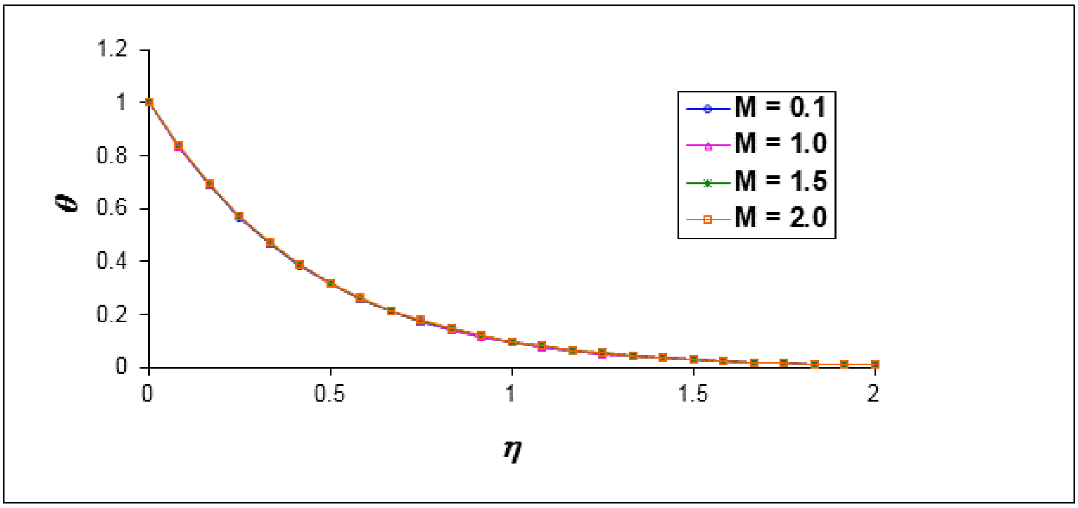

Figure 4.2.18: Temperature profiles for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $f_w = 3.0$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

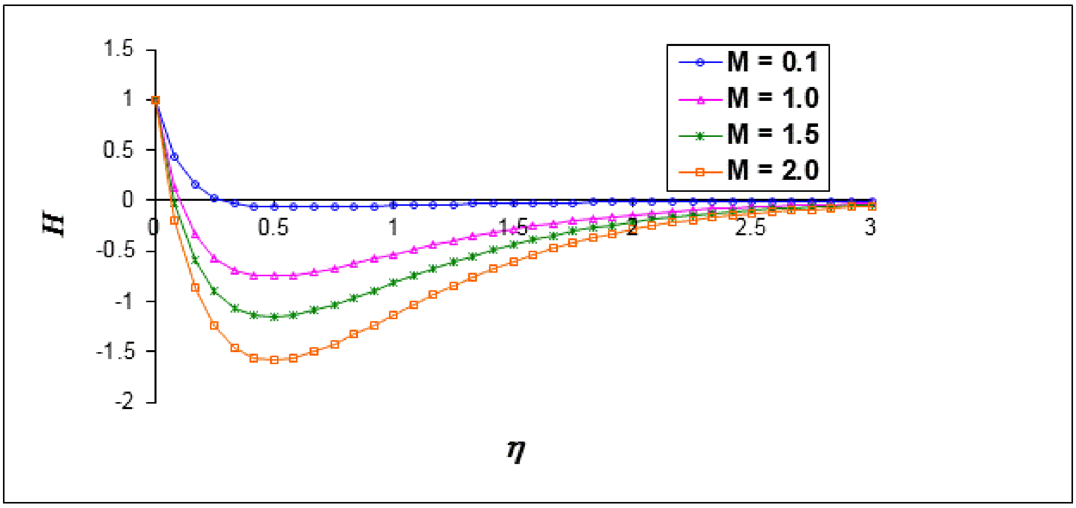

Figure 4.2.19: Induced magnetic field distribution for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $f_w = 3.0, G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

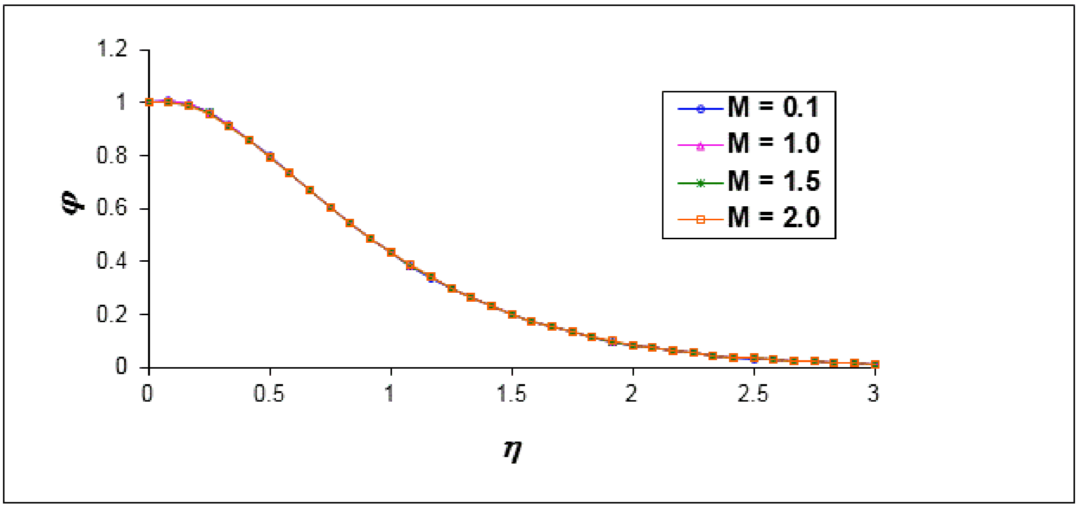

Figure 4.2.20: Concentration distribution for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 2.0$, $S_c = 0.6$, $f_w = 3.0$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

Displayed Figure 4.2.21 exhibits the effect of Prandtl number (P_r) on the non-dimensional velocity profiles. We observed that the velocity decreases significantly with the increase of P_r within the boundary layer. This is due to fact that as P_r increases, the dynamic viscosity of the fluid increases which then slow down the velocity of the fluid. Figure 4.2.22 illustrates the effects of Prandtl number, P_r on the non-dimensional temperature profiles within the boundary layer. As we observed in the figure, the temperature abruptly decreases when the values of P_r increase at a fixed value of η, so that for a higher Prandtl number fluid, the thermal conductivity is relatively lower, which reduces conduction as well as the thermal boundary layer thickness and finally temperature is reduced. Therefore increasing P_r is to increase the heat transfer rate at the surface and as a result the temperature gradient at the surface increases.

Figure 4.2.23 and Figure 4.2.24 show the effect of P_r on induced magnetic field and mass concentration, respectively.

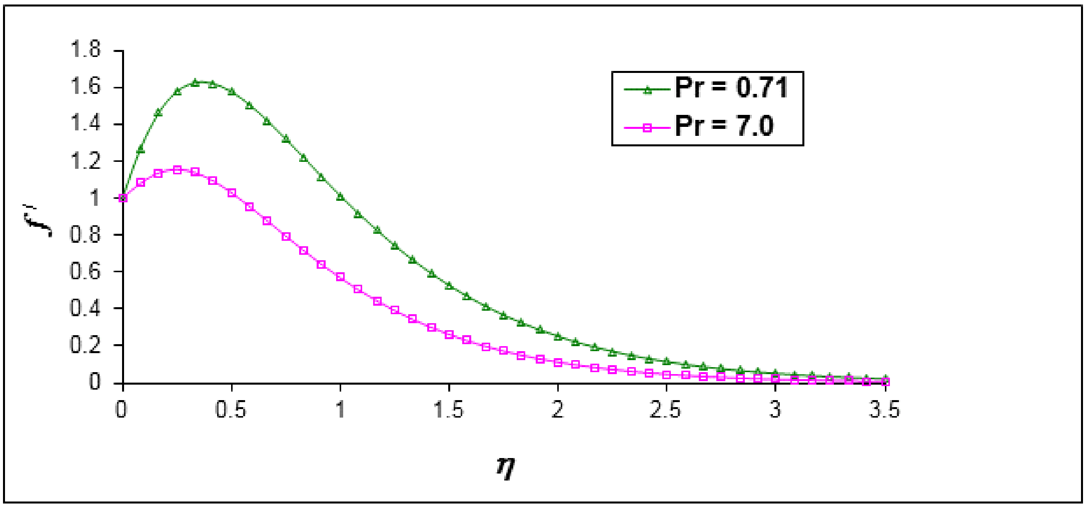

Figure 4.2.21: Velocity profiles for different values of P_r (with fixed values of $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $f_w = 3.0$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

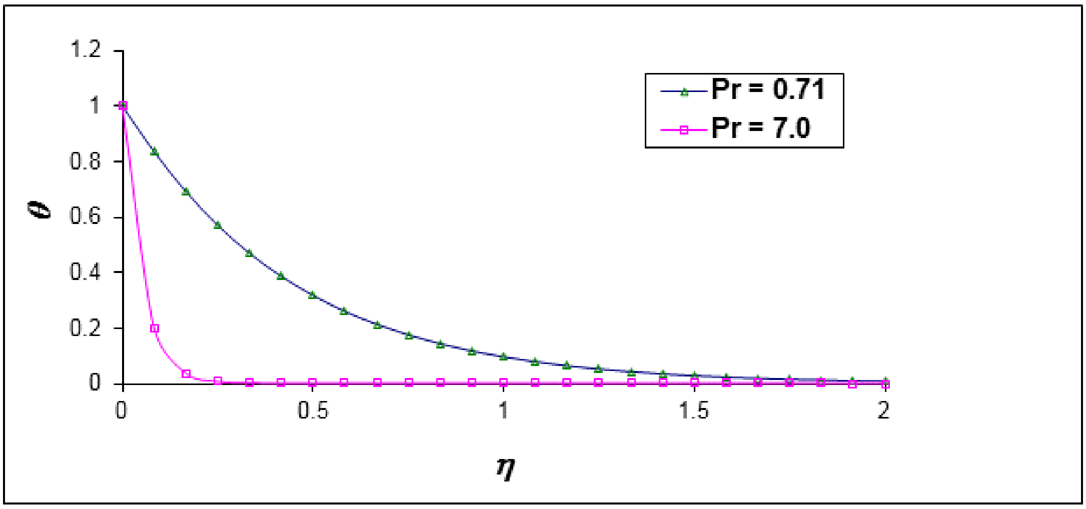

Figure 4.2.22: Temperature profiles for different values of P_r (with fixed values of $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $f_w = 3.0$, $G_r = 6.0$ $G_m = 4.0$ and $\alpha = 45^0$).

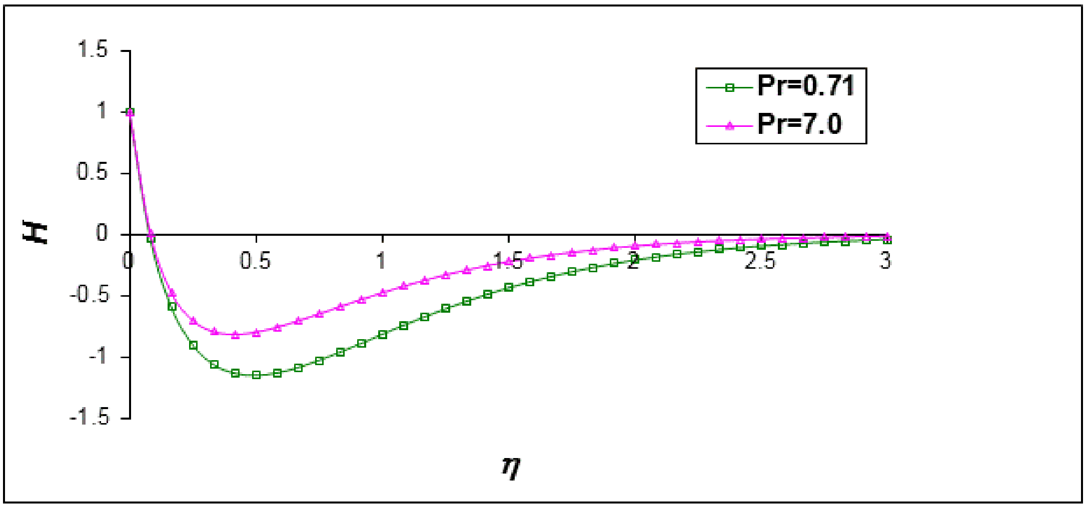

Figure 4.2.23: Induced magnetic field distribution for different values of P_r (with fixed values of $S_0 = 2.0$, $S_c = 0.6$, $M = 1.5$, $f_w = 3.0$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

Here again the magnitude of the induced magnetic fields decrease with the increasing values of P_r as is seen in Figure 4.2.23. Also with increasing P_r, the concentration is found to increase firstly very close to the plate surface and after that it further decreases and asymptotically approaches to zero away from the plate surface (Figure 4.2.24).

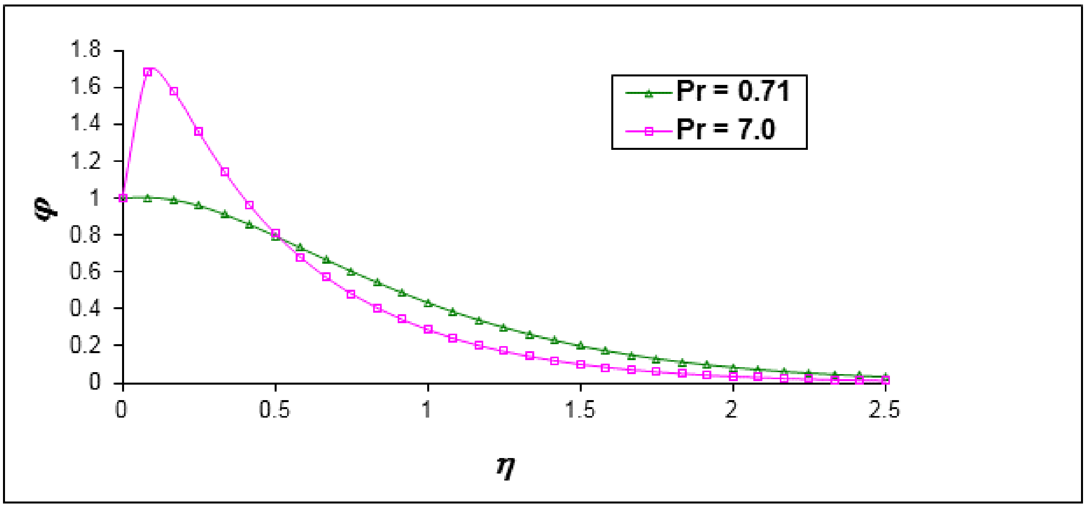

Figure 4.2.24: Concentration distribution for different values of P_r (with fixed values of $S_0 = 2.0, S_c = 0.6, M = 1.5, f_w = 3.0, G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

Finally, the dependency of wall shear stress (in terms of local skin-friction coefficient (C_f)), wall heat flux (in terms of Nusselt number (N_u)) and wall mass flux (in terms of Sherwood number (S_h)), which are of physical interests, on the concerned non-dimensional parameters and numbers have been observed. The variation of values proportional to C_f in term of shear stress$\left(f''(0)\right)$,N_u in term of wall heat transfer$\left(-\theta'(0)\right)$and S_h in term of wall mass transfer $\left(-\varphi'(0)\right)$ with the variation of the values of different selected established dimensionless parameters and numbers are tabulated and have been illustrated through Tables 4.2.1 to 4.2.7.

Table 4.2.1: Variations of the values proportional to the coefficients of skin friction $\left(f''(0)\right)$, rate of heat transfer $\left(-\theta'(0)\right)$and rate of mass transfer$\left(-\varphi'(0)\right)$with the variation of S_0 (for fixed values of $P_r = 0.71, S_c = 0.6, f_w = 3.0, M = 1.5, G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

S_0	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
0.0	0.811797	2.009	2.175000
1.5	2. 811797	2.009	0.507300
2.0	3.478463	2.009	-0.048962
5.0	7.478463	2.009	-3.384906

Form Table 4.2.1, it is observed that with the increase in S_0, the coefficient of skin friction increases and the rate of mass transfer decreases. No effect of S_0 on the coefficient of heat transfer is perceived.

Table 4.2.2: Variations of the values proportional to the coefficients of skin friction $\left(f''(0)\right)$, rate of heat transfer $\left(-\theta'(0)\right)$and rate of mass transfer$\left(-\varphi'(0)\right)$with the variation of f_w (for fixed values of $P_r = 0.71, S_c = 0.6, M = 1.5, G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

f_w	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
1.5	12.517590	-0.708820	0.708820
3.0	3.478463	0.048962	-0.048962
5.0	-1.537190	0.645236	-0.645236

From Table 4.2.2, it is seen that, with the increase of f_w, the coefficient of skin friction and rate of mass transfer highly decreased but the rate of heat transfer increased. The usual stabilizing effect of the suction parameter on the boundary layer growth is also evident from this Table.

Table 4.2.3: Variations of the values proportional to the coefficients of skin friction $\left(f''(0)\right)$, rate of heat transfer $\left(-\theta'(0)\right)$ and rate of mass transfer $\left(-\varphi'(0)\right)$ with the variation of G_r (for fixed values of $P_r = 0.71, S_c = 0.6, f_w = 3.0, M = 1.5, G_m = 4.0$ and $\alpha = 45^0$).

G_r	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
-4.0	-1.216370	2.009	-0.048962
-2.0	-0.277410	2.009	-0.048962
2.0	1.600529	2.009	-0.048962
4.0	2.539496	2.009	-0.048962
6.0	3.478463	2.009	-0.048962

Table 4.2.3 shows that with the increase of G_r, the coefficient of skin friction gradually increases but both the rate of heat and mass transfers remain unaffected as G_r varies.

Table 4.2.4: Variations of the values proportional to the coefficients of skin friction $\left(f''(0)\right)$, rate of heat transfer $\left(-\theta'(0)\right)$ and rate of mass transfer $\left(-\varphi'(0)\right)$ with the variation of M (for fixed values of $P_r = 0.71, S_c = 0.6, f_w = 3.0, G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

M	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
0.1	4.468413	2.15419	-0.223190
1.0	3.832017	2.06884	-0.120722
1.5	3.478463	2.00900	-0.048962
2.0	3.124910	1.94029	0.033500

Form Table 4.2.4 it is seen that, as the magnetic parameter (M) increases, the coefficient of skin friction and the rate of heat transfer decreased slightly, but the rate of mass transfer diminutively increased.

Table 4.2.5: Variations of the values proportional to the coefficients of skin friction $(f''(0))$, rate of heat transfer $(-\theta'(0))$ rate of mass transfer $(-\varphi'(0))$ with the variation of P_r (for fixed values of $S_c = 0.6$, $f_w = 3.0$, $M = 1.5$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

P_r	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
0.71	3.478463	2.0090	-0.048962
5.0	1.061562	13.4510	-13.59124
7.0	0.947276	18.7342	-19.91232

It is observed form Table 4.2.5 that, with the increase of the Prandtl number P_r, the coefficient of skin friction decreases. Also the rate of heat transfer highly increases whereas the rate of mass transfer extensively decreases as P_r increases.

Table 4.2.6: Variations of the values proportional to the coefficients of skin friction $(f''(0))$, rate of heat transfer $(-\theta'(0))$ and rate of mass transfer $(-\varphi'(0))$ with the variation of S_c (for fixed values of $P_r = 0.71$, $f_w = 3.0$, $M = 1.5$, $G_r = 6.0$, $G_m = 4.0$ and $\alpha = 45^0$).

S_c	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
0.1	14.48957	2.009	-0.003342
0.6	3.478463	2.009	-0.048962
0.7	3.161003	2.009	-0.061487
1.5	2.14513	2.009	-0.179643
5.0	1.522908	2.009	-0.796665

Table 4.2.6 shows that with the increase of S_c, the coefficient of skin friction decreases sharply but the rate of mass transfer evenly decreases. No through effect of S_c on the rate of heat transfer is seen here.

Table 4.2.7: Variations of the values proportional to the coefficients of skin friction $(f''(0))$, rate of heat transfer $(-\theta'(0))$ and rate of mass transfer $(-\varphi'(0))$ with the variation of G_m (for fixed values of $P_r = 0.71$, $f_w = 3.0$, $M = 1.5$, $S_c = 0.6$, $G_r = 6.0$ and $\alpha = 45^0$).

G_m	$f''(0)$	$-\theta'(0)$	$-\varphi'(0)$
-4.0	-6.29931	2.009	-0.048962
4.0	3.478463	2.009	-0.048962

The coefficient of skin friction remarkably decreases with decreasing values of G_m from positive to negative but no consequence of G_m on the rate of heat and mass transfers are observed as is seen in Table 4.2.7.

CHAPTER V

Conclusions

In the present research, a flow model of steady natural convection heat and mass transfer of viscous incompressible electrically conducting fluid past a semi-infinite vertical electrically non-conducting moving porous plate has been studied considering the effects of thermal diffusion and uniform inclined magnetic field. Numerical approximation of the second order solutions in collaboration with the first order solutions has been carried out using the perturbation technique. The influence of inclined magnetic field on the independent field variables have been investigate considering three different inclined angles of uniform applied magnetic field, namely, 0^0, 45^0 and 75^0. From the investigation of the present problem, some conclusions have been made as follows:

- The increase of inclined magnetic field angle from 0^0 to 75^0 tends to slow down the velocity field. The magnitude of the induced magnetic field decreases highly with the increase of inclined angle. But the mass concentration process slow down a little only inside the boundary layer with the increase of inclined angle while temperature profiles are not affect by any means with the variation of the inclined angle.
- The velocity, mass concentration and the magnitude of the induced magnetic field increases with the increase of Soret number but there is no remarkable effect of the number on the temperature field is observed.
- The velocity, temperature and magnitude of induced magnetic fields decreases quickly with the increase of the suction parameter. Concentration shows increasing-decreasing characteristics as the suction values increases.
- The fluid velocity, temperature and magnitude of induced magnetic field decrease highly with the increase of Prandtl number. But the concentration has increasing-decreasing behavior with the increasing values of the Prandtl number.
- The local skin-friction coefficient increases with the increase of Soret number, Grashof number and modified Grashof numberbut decrease with the increase of suction parameter, magnetic parameter, Prandlt number and Schmidt number.
- Nusselt number increases with the increase of suction parameter and Prandlt number but decreases with the increase of magnetic parameter. Whereas it remains unchanged with increasing Soret number, Grashof number and modified Grashof number and Schmidt number.

- Sherwood number increases only with the increase of Magnetic parameter but decreases with the increase of Soret number, Suction parameter, Prandlt number and Schmidt number. Whereas it remain unaffected with the increase of Grashof number and modified Grashof number.

REFERENCES

Agrawal, H.L., Ram, P.C. and Singh, V.(1983). “Heat and mass transfer of an oscillatory flow with hall current, II.” Astrophys.Space Sci., 94, 383–393.

Ahmed, T., Alam Md.M. (2014). “Chemically reacting ionized fluid flow through a vertical plate withinclined magnetic field in rotating system.” Procedia Engineering 90, 301–307 (Earlier presented in 10th International Conference on Mechanical Engineering, ICME 2013).

Ahmed, S. and Chamkha, A. J. (2010). “Effects of chemical reaction, heat and mass transfer and radiation on MHD flow along a vertical porous wall in the presence of induced magnetic field.” Int. Journal of Industrial Mathematics, 2, 245–261.

Ahmed, S. and Zueco, J. (2010). “Combined heat and mass transfer by mixed convection MHD flow along a porous plate with chemical reaction in presence of heat source.” Appl. Math.and Mech., 31, 1217–1230.

Ahmed, S., Bég, O. A.,Vedad,S.,Zeinalkhani, M. and Heidari, A. (2012). “Mathematical modelling of magnetohydrodynamic transient free and forced convective flow with induced magnetic field effects.” Int. Pure and Appl. Sci. and Tech., 11, 109–125.

Alam, Md. M. (1995). “Steady MHD free convection and mass transfer flow with thermal diffusion and large suction.” Ph. D. Thesis, Ch 7, 134.

Alam M. S., ALi, M. and Hossain, Md. D. (2013). “Heat and Mass Transfer in MHD free convection flow over an inclined plate with Hall current.”Volume 2, 81–88.

AlamM.S.,Ali M, Alim M.A., Saha A. (2014). “Steady MHD boundary free convective heat and mass transfer flow over an inclined porous plate with variable suction and Soret effect in presence of hall current.” Banglasesh J. Sci. Ind. Res. 49 (3), 155–164.

Alam, Md. M., Islam, M. R. and Rahman, F. (2008). “Steady heat and mass transfer by mixed convection flow from a vertical porous plate with induced magnetic field, constant heat and mass fluxes.” Thammasat Int. J. Sc. Tech., 13, 1–13.

Alam, M.S. Rahman, M. M. Ferdows, M. M., Kaino, M., Koji, Eunice, M. and Postelnicu, A. (2007). “Diffusion-thermo and thermal-diffusion effects on free convective heat and mass transfer flow in a porous medium with time dependent temperature and concentration.” Int. J. Appl. Engng. Res., 2(1), 81–96.

Alfven, H. (1942). “On the existence of electromagnetic Hydromagnetic waves.” Mat.Astro.Fysik.Bd., 2, 295.

Ali, F. M., Nazar, R., Arifin, N. M. and Pop, I. **(**2011). “MHD Mixed Convection Boundary Layer Flow Toward a Stagnation Point on a Vertical Surface With Induced Magnetic Field, Trans.” ASME Journal of Heat Transfer, 133, 1–6.

Asaduzzaman, Md., Islam, Md.R. and Islam, A.(2016). "Transient heat transfer flow along a vertical plate with induced magnetic field." International Journal of Scientific & Engineering Research, 7 (11), 10–22.

Bég, O.A., Bakier, A. Y., Prasad, V.R. Zueco, J. and Ghosh, S. K. (2009). "Non-similar, laminar, steady, electrically-conducting forced convection liquid metal boundary layer flow with induced magnetic field effects." Int. J. Thermal Sciences, 48, 1596–1606.

Bestman, A.R. (1990). "The boundary-layer flow past a semi-infinite heated porous plate for a two-component plasma." Astrophysics and Space Science, 173 (1),93–100.

Caldwell, D.R. (1974). "Experimental studies on the onset of thermohaline convection." J. Fluid Mech., 64, 347–367.

Choudhary, R.C. and Sharma, B.K. (2006). "Combined heat and mass transfer by laminar mixed convection flow from a vertical surface with induced magnetic field." J. Appl. Phys., 99, 034901–10.

Cobble, M.H. (1977). "Manetofluiddynamic flow with a pressure gradient and fluid injection." 11,249-256.

Cramer, K.R. and Pai, S.I. (1973). "Magneto fluid Dynamics for Engineers and applied physicists." McGraw Hill, New York.

Elshehawey, E.F., El Elbarbary, M.E. and Elgazery, N.S. (2003). "Effect of inclined magnetic field on magneto fluid flow through a porous medium between two inclined wavy porous plates (numerical study)." Applied Mathematics and Computation, 135 (1), 85–103.

Farady, M. (1832). "Electromagnetic Force Field, Particle/Field Duality." 2, 525–1333.

Gebhart, B. and Pera, L. (1971). "The nature of vertical convection flows resulting from the combined buoyancy effects of thermal and mass diffusion." Int.J. Heat Mass Transfer, 14, 2025.

Georgantopolous, G. A.,Nanousis, N. D. and Goudas, C. L. (1979). "Effects of mass transfer on the free convection flow in the Stokes' problem for an infinite vertical limiting surface." Astrophysics and Space Science, 66 (1), 13–21.

Groots, S. R. T. and Mozur, P. (1962). "Non-equilibrium Thermodynamics." North Holland, Amsterdam.

Gundagani, M., Sheri, S., Paul, A. and Reddy, M.C.K. (2013). "Unsteady magnetohydrodynamic free convective flow past a vertical porous plate." International Journal of Applied Science and Engineering, 11(3), 267–275.

Hossain M.M.T. and Khatun, M. (2012). “Study of Diffusion-Thermo Effect on Laminar Mixed Convection Flow and Heat Transfer from a Vertical Surface with Induced Magnetic Field.” Int. J. of Appl. Math and Mech., Vol. 8(5), 40–60.

Hossain, M.M.T., Zaman, M.A., Rahman, F. and Hossain, M.A. (2013). “Steady MHD free convection heat and mass transfer flow about a vertical porous surface with thermal diffusion and induced magnetic field.” American Institute of Physics (AIP) Conf. Proc., 1557, 594–603; doi: 10.1063/1.4824172;© 2013 AIP Publishing(http://dx.doi.org/10.1063/1.4824172).

Islam, A., Islam, M.M.,Rahman, M., Ali, L.E. and Khan, Md. S. (2016). “Unsteady heat transfer of viscous incompressible boundary layer fluid flow through a porous plate with induced magnetic field.” Journal of Applied Mathematics and Physics, 4, 294–306 (http://dx.doi.org/10.4236/jamp.2016.42037).

Kafoussias, N. G. (1992). “MHD thermal-diffusion effects on free convective and mass transfer flow over an infinite vertical moving plate.” Astrophysics and Space Science, 192, 11–19.

Kay, J. M (1953). “Boundary layer flow alonga flat plate with uniform suction.” Cam.Univ. Engng.Lab., 2628.

Khan, Md.S., Wahiduzzaman, M., Karim, I., Islam, Md.S. and Alam, Md.M. (2014). “Heat generation effects on unsteady mixed convection flow from a vertical porous plate with induced magnetic field.” (10th International Conference on Mechanical Engineering, ICME 2013), Procedia Engineering, 90, 238–244.

Kim, Y. J., (2004). “Heat and mess transfer in MHD micro polar flow over a vertical moving porous plate in a porous medium.” Transport in Porous Media, 56(1), 17–37.

Legros, J. G., Van Hook, W. K. and Thomas, G. (1968). Chem. Phys. Lett., 2, 696.

Mohammad, F., Masatiro, O., Abdus, S. and Mohamud, A. (2005). “Similarity solutions for MHD flow through vertical porous plate with suction.” Journal of Computational and Applied Mechanics, 6(1), 15–25.

Nadeem, S. and Akram, S. (2010). “Influence of inclined magnetic field on peristaltic flow of a Williamson fluid model in an inclined symmetric or asymmetric channel.” Mathematics Computing Modeling, 52 (1/2),107–119.

Nanbu, K. (1971). “Vortex flow over a flat surface with suction.” AIAA, J. 9 (8), 1642–1643.

Nanousis, N. (1992). “Thermal-diffusion effects on MHD free convection and mass transfer past a moving infinite vertical plate in a rotating system.” Astrophysics and Space Science, 191, 313–322.

Opiyo, R.O., Alfred, W.M. and Jacob, K.B. (2017). “Numerical computation of steady buoyancy driven MHD heat and mass transfer past an inclined infinite flat plate with sinusoidal surface boundary conditions.” Applied Mathematical Sciences, 11(15), 711–729 (https://doi.org/10.12988/ams.2017.7127).

Patanker, S. V. and Spalding, D. B. (1970). “Heat and Mass Transfer in Boundary Layers.” 2ndEdn.Intertext Books, London.

Pantokratoras, A. (2007). Comment on “Combined heat and mess transfer by laminar mixed convection flow from a vertical surface with induced magnetic field”.Journal of Applied Physics, 102, 076113.

Postelnicu, A. (2004). “Influence of magnetic field on heat and mass transfer by natural convection from vertical surfaces in porous media considering Soret and Dufour effects.” Int. J. Heat and Mass Transfer, 47, 1467–1472.

Rani, K. J., Reddy, G. V. R., and Murthy, Ch. V. R. (2015). “Heat and mass Transfer effects on MHD free convection flow over an inclined plate embedded in a porous medium.” 13(4),1998–2016.

Raptis, A. (1986). “Flow through a porous medium in the presence of magnetic field.” Int. J. Energt Res., 10, 97–101.

Raptis, A. and Kafoussias, N.G. (1982). “Magnetohydrodynamic free convection flow and mass transfer through porous medium bounded by an infinite vertical porous plate with constant heat flux.” Can. J. Phys., 60(12), 1725–1729.

Raptis, A. and Shing A. K. (1983). “MHD Free convection flow past an accelerated vertical plate.International Communication in Heat and Mass Transfer.” 10 (4), 313–321.

Reddy, V.P., Kumar, R.V.M.S.S. K., Reddy, G.V., Prasad, P.D. and Varma, S.V.K. (2015). “Free convection heat and mass transfer flow of chemically reactive and radiation absorption fluid in an aligned magnetic field.” (International Conference on Computational Heat and Mass Transfer-2015), Procedia Engineering, 127, 575–582.

Reddy, B. P. and Rao, J. A. (2011). “Numerical solution of thermal diffusion effect on an unsteady MHD free convective mass transfer flow past a vertical porous plate with Ohmic dissipation.” Int. J. of Appl. Math.and Mech., 7 (8), 78–97.

Rosenberg, D. U. V. (1969). "Method for numerical solutions of partial differential equations." American Elsevier, New York.

Sattar, M. A., Alam, M. S. and Rahman, M. M. (2006). "MHD free convective heat and mass transfer flow past an inclined surface with heat generation." Thammasat Int. J. Sc. Tech. ll (4), 1–8.

Schlichting, H. (1968). "Boundary Layer Theory." McGraw-hill, New York.

Singh, A. K. (1980). "Hydromagnetic boundary-layer flows with large suction." 115(2),387–391.

Singh, A.K. and Dikshit, G.K. (1988). "Hydromagnetic flow past a continuously moving semi-infinite plate for large suction." Astrophysics and Space Science, 148 (2), 249–256.

Singh, N. K., Kumar, V. and Sharma, G. K. (2016). "The Effect of Inclined Magnetic Field on Unsteady Flow Past on Moving Vertical Plate with Variable Temperature." IJLTEMAS, V (II), 34–37.

Somers, F.V. (1956). J. Appl, mech., 23, 295.

Soundalgekar, V. M. and Ramanamurthy, T. V. (1980). "Heat transfer flow past a continuous moving plate with variable temperature." *Warme-Und Stoffubertrag*, 14,91–93

Spalding D. B. (1977). "GENMIX, a general computer program for two dimensional parabolic phenomena." Pergamon Press, Oxford, Uk.

Wahiduzzaman, M., Biswas, R., Eaqub, Md. A., Khan, Md. S. and Karim, I. (2015). "Numerical solution of MHD convection and mass transfer flow of viscous incompressible fluid about an inclined plate with hall current and constant heat flux." Journal of Applied Mathematics and Physics, 3, 1688–1709 (http://dx.doi.org/10.4236/jamp.2015.312195).