Generalization of Distributive Lattice with Pseudo complementation

by

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Dedicated

To

My Parents

Declaration

This is to certify that the thesis work entitled "**Generalization of Distributive Lattice with Pseudo-complementation**" has been carried out by Shubhra Paul, in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for award of any degree or diploma.

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Shubhra Paul

INTRODUCTION

Lattice theory is an important part of Mathematics. Distributive lattices with Pseudo complementation have played many roles in development of lattice theory. Historically, lattice theory started with Boolean distributive lattices: as a result, the theory of distributive lattices is the most extensive and most satisfying chapter in the history of lattice theory. Distributive lattices have provided the motivation for many results, in general lattice theory. Many conditions on lattices and on element and ideals of lattices are weakened forms of distributivity is imposed on lattices arising in various areas of mathematics, especially algebra.

In lattice theory there are different classes of lattices known as variety of lattices. Class of Boolean lattice is of course the most powerful variety. Throughout this thesis we will be concerned with another large variety known as the class of distributive Pseudo complemented lattice have been studied by several authors [1],[2],[3],[4],[5],[6].

On the other hand extended the notion of Pseudo complementation for meet semi lattices.

There are two concepts that we should be able to distinguish: a lattice $< L,\wedge,\vee >$, in which every element has a Pseudo complement and an algebra, $< L,\wedge,\vee,*,0,1 >$ where $< L,\wedge,\vee,0,1 >$ is a bounded lattice and where, for every $a \in L$, the element a^* is a Pseudo complement of a. We shall call the former a Pseudo complemented lattice and the later a lattice with Pseudo complementation (as an operation).

The realization of special role of distributive lattices moved to break with the traditional approach to lattice theory, which proceeds from partially ordered sets to general lattices, semi modular lattices, modular lattices and finally distributive lattices.

In order to review, we include definitions, examples, solved problems and proof of some theorems. This work is divided into four chapters.

Chapter-one is a prelude to the main text of the thesis, related to poset and various types of lattices, such as sublattice, ideal of lattice, bounded lattice, complete lattice.

In chapter two we have discussed "Modular and distributive lattice" and this chapter is the concept of this work. Here we study the definition and examples of modular and distributive lattice. Some important theorem like "A modular lattice L is distributive if it has no sublattice isomorphic diagonal lattice M_5 ". Every modular lattice is distributive but converse is not true.

The next chapter we discuse "Prime ideal of a lattice", "Minimal prime ideal" and "Minimal prime n-ideal".

Chapter four dealt with the Distributive lattices with Pseudo complementation. This is the main part of my work. In this chapter we have discussed some definitions and some important theorems like "Any complete lattice that satisfies the Join Infinite Distributive (JID) identity is a Pseudo complemented distributive lattice."

CONTENTS

CHAPTER I

PRELIMINARIES

1.1 Some Definitions of Lattices:

Definition 1.1.1. Let A and B be two sets and from the set $A \times B$ of all ordered pairs (a,b) with $a \in B$. If $A=B$, we write A^2 for $A \times A$. Then a binary relation R on A can simply be defined as a subset of A^2. The elements a,b $(a,b \in A)$ are in relation with respect of R if $(a,b) \in R$. For $(a,b) \in R$, we will also write "$a\ R\ b$ or $a \equiv b(R)$" and as "a is related to b by R ".

Definition 1.1.2. A non-empty set P together with a binary relation R is said to be a partially ordered set or a poset if the following conditions holds:

For all $a,b,c \in P$ we have

(P1) $a\ R\ a$ i.e P is reflexive

(P2) $a\ R\ b$ and $b\ R\ a$ imply that $a = b$ i.e P is anti-symmetric

(P3) $a\ R\ b$ and b R c imply that $a\ R\ c$ i.e P is transitive.

For convenience, we generally use the symbol $\leq$ in place of R. Thus whenever we say that P is a poset, it would be understood that $\leq$ is the relation defined on P, unless another symbol is mentioned.

Examples 1.1.3. (i) Let X be any set,then $(P(x), \subseteq)$ is a poset.

(ii) Let N be the set of natural numbers under the usual $\subseteq$ is a poset.

(iii) The integers, rationals and real numbers also from posets under usual $\subseteq$

Definition 1.1.4. A poset $(A\ ;\ \leq)$ is called a chain if it satisfies the following condition (P2) $a \leq b$ or $b \leq a$, $\forall a, b \in A$. (linearity)

Remark 1.1.5. A chain is also known as a totally ordered set or a toset on a linearity ordered set.

Definition 1.1.6. Let $(P, \leq)$ be a poset and $a,b \in P$, Then a and b are comparable if $a \leq b$ or $b \leq a$, otherwise a and b are incomparable in notation $a \parallel b$.

Remark 1.1.7. A chain is therefore a poset in which there be no incomparable element.

Definition 1.1.8. Let Φ be a relation defined on a set X.Then converse of Φ (denoted by $\overline{\Phi}$) is defined by $a\ \overline{\Phi}\ b \Leftrightarrow b\ \Phi\ a$; $a, b \in X$.

If (X,Φ) be a poset then the poset $(\bar{X},\overline{\Phi})$ where $\bar{X}= X$ and $\overline{\Phi}$ is converse of Φ is called dual of X.

Theorem 1.1.9. If a set X is from a poset under a relation Φ, then X from a poset under $\overline{\Phi}$, the converse of Φ.

Proof:

$a\ \overline{\Phi}\ a$ as $a\ \Phi\ a$ for all $a \in X$ shows $\overline{\Phi}$ is reflexive.

Let $a\ \overline{\Phi}\ b$ and $b\ \overline{\Phi}\ a$ then $b\ \Phi\ a$ and $a\ \Phi\ b$ i.e $a\ \Phi\ b$ and $b\ \Phi\ a \Longrightarrow a = b$.

Thus $\overline{\Phi}$ is anti –symmetric.

Let $a\ \overline{\Phi}\ b$, $b\ \overline{\Phi}\ c$ then $b\ \Phi\ a$, $c\ \Phi\ b$

Or, $c\ \Phi\ b$, $b\ \Phi\ a$

Or, $c\ \Phi\ a \Longrightarrow a\ \Phi\ c$

Or that $\overline{\Phi}$ is transitive and hence is a partial ordering. ■

Remark 1.1.10. We will use the notations $a \wedge b = Inf\{a,b\}$ and $a \vee b = Sup\{a,b\}$ and call $\wedge$, the meet and $\vee$, the joint. In lattice, they are both binary operations which means that they can be applied to a pair of elements of L. Thus $\wedge$ a map of L^2 into L, and so $\vee$.

Example 1.1.11. (i) Let X be a non-empty set. Then $P(X)$ the power set of X under " contain in " $\subseteq$ Relation from a poset and this poset $(P(X),\subseteq)$ is a lattice.Here for $A,B \in P(X)$ [A, B are subset of X].

$$A \wedge B = A \cap B \text{ and } A \vee B = A \cup B.$$

As a particular case, when $X=\{a,b\}$
$P(X)= \{\emptyset,\{a\},\{b\},\{a,b\}\}$
Then $(P(X), \subseteq)$ is represented by the following figure 1.1

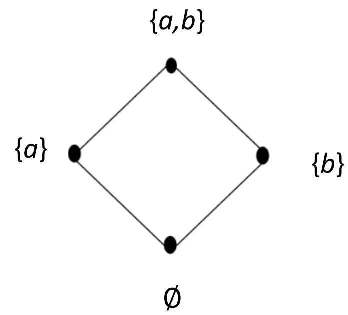

Fig. 1.1

(ii) Let the set $X=\{1,2,4,5,10,20\}$ of the factors of 20 under divisibility forms a lattice. It is represented by the following figure 1.2. When the particular ordering relation is divisibility that is, when we define the relation as $a \leq b$ iff a/b. Then $a \wedge b = g.l.b(a,b)$ and $a \vee b = l.u.b(a,b)$. By definition $a \wedge b = Inf\{a,b\}$ and if $Inf(a,b)=x$, then we should have $x \leq a$, $x \leq b$ and if $y \leq a$, $y \leq b$ then $y \leq x$. Which implies that x/a, x/b and if y/x. Now by definition $g.l.b\ (a,b)= c$ means c/a, c/b and if d/a, d/b then d/c. Therefore we have $g.l.b\ (a,b)$. Similarly $l.u.b\ (a,b)= a \vee b$.

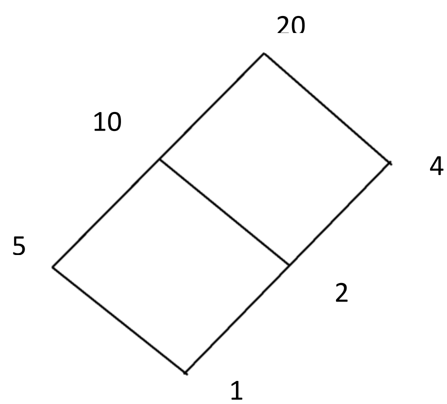

Fig. 1.2

Theorem 1.1.12. For any a and b in a lattice $(L,\leq)$, $a \leq a \vee b$, $a \wedge b \leq a$

Proof:

Since the join of a and b is an upper bound of L, hence $a \leq a \vee b$. Since the meet of a and b is a lower bound of L, hence $a \wedge b \leq a$. ■

Theorem 1.1.13. If L is any lattice then for any $x,y,z \in L$, the following results hold.

(L1) $x \wedge x = x$, $x \vee x = x$ (Idem potency)

(L2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ (Commutative)

(l3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

$x \vee (y \vee z) = (x \vee y) \vee z$ (Associativity)

(L4) $x \wedge (x \vee y)= x$

$x \vee (x \wedge y)= x$ (Absorption identities) ■

Theorem 1.1.14.

(i) Let the poset $L=(L;\leq)$ be a Lattice. Set $e_1 \wedge e_2=Inf\{e_1, e_2\}$, $e_1 \vee e_2=Sup\ \{e_1, e_2\}$.

Then the algebra $L^a=(L; \wedge, \vee)$ is a lattice.

(ii) Let the algebra $L=(L; \wedge, \vee)$ be a lattice. Set $e_1 \leq e_2$ iff $e_1 \wedge e_2= e_1$. Then $L^p=(L;\leq)$

is a poset and the poset L^p is a lattice.

(iii) Let the poset $L=(L; \wedge, \vee)$ be a lattice, then $(L^a)^p=L$.

(iv) Let the algebra $L=(L; \wedge, \vee)$ be a lattice, then $(L^p)^a=L$.

Proof:

(i) Since $L=(L;\leq)$ be a lattice, so $e_1 \wedge e_2= Inf\{e_1, e_2\}$ and $e_1 \vee e_2= Sup\ \{e_1, e_2\}$ exist in L.

Now $\forall\ e_1 \in L$, $e_1 \wedge e_2= Inf\{e_1, e_2\}=e_1$ and $\forall\ e_1 \in L$, $e_1 \vee e_2= Sup\{e_1, e_2\}=e_1$. i.e idempotent law is satisfied.

For $e_1, e_2 \in L$, $e_1 \wedge e_2= Inf\{e_1, e_2\}= Inf\{e_2, e_1\}=e_2 \wedge e_1$

Similarly $e_1 \vee e_2= e_2 \vee e_1$

i.e satisfies the commutative law $\forall e_1. e_2. e_3 \in L$,

$e_1 \wedge(e_2 \wedge e_3) = Inf\{e_1\ Inf\ \{e_2, e_3\}\} = Inf\{\mathrm{Inf}\{e_1, e_2\}, e_3\ \} = (e_1 \wedge e_2) \wedge e_3$

Similarly, $e_1 \vee(e_2 \vee e_3) = (e_1 \vee e_2) \vee e_3$

i.e so it has the associate property.

Finally for all $e_1, e_2 \in L$, $e_1 \wedge (e_2 \vee e_3)$= $Inf\{e_1, Sup\{e_1, e_2\}\}$ = e_1

Similarly $e_1 \vee (e_2 \wedge e_3)$= $Sup\{e_1, Inf\{e_1, e_2\}\}$ = e_1

Which is absorption law.

Therefore L^a=$(L; \wedge, \vee)$ is a lattice.

(ii) Here algebra L=$(L; \wedge, \vee)$ is a lattice.

Now for $e_1, e_2 \in L$, $e_1 \leq e_2$ iff $e_1 = e_1 \wedge e_2$, Clearly ,"$\leq$" is reflexive as $\wedge$ is idempotent.

Suppose, $e_1 \leq e_2$ and $e_2 \leq e_1$, then $e_1 = e_1 \wedge e_2$ and $e_2 = e_2 \wedge e_1$.

Thus $e_1 = e_2$ as $\wedge$ is commutative

Hence $\leq$ is anti-symmetric.

Now, let $e_1 \leq e_2$ and $e_2 \leq e_3$, then $e_1 = e_1 \wedge e_2$ and $e_2 = e_2 \wedge e_3$.

Thus $e_1 = e_1 \wedge e_2$

$= e_1 \wedge (e_2 \wedge e_3)$

$= (e_1 \wedge e_2) \wedge e_3$ [as $\wedge$ is associative]

$= e_1 \wedge e_3$

Then $e_1 \leq e_3$ and so $\leq$ is transtitive.

Therefore $(L;\leq)$ is a poset.

Now $(e_1 \wedge e_2) \wedge e_1 = e_1 \wedge (e_2 \wedge e_1)$

$= e_1 \wedge (e_1 \wedge e_2)$

$= (e_1 \wedge e_1) \wedge e_2$

$= e_1 \wedge e_2$

And $(e_1 \wedge e_2) \wedge e_2 = e_1 \wedge (e_2 \wedge e_2) = e_1 \wedge e_2$

Therefore, $e_1 \wedge e_2 \leq e_1$ and $e_1 \wedge e_2 \leq e_2$

Now suppose $x \leq e_1, x \leq e_2$ for some $x \in L$

Then $x = x \wedge e_1$, $x = x \wedge e_2$

Thus $x = (x \wedge e_1) = (x \wedge e_1) \wedge e_2 = x \wedge (e_1 \wedge e_2)$

This implies $x \leq e_1 \wedge e_2$ and so $e_1 \wedge e_2 = Inf\{e_1, e_2\}$

Hence $Inf\{e_1, e_2\}$ exists in L.

Finally, for $e_1, e_2 \in L$,

$e_1 \wedge (e_1 \vee e_2) = e_1$ and $e_2 \wedge (e_1 \vee e_2) = e_2$ (by absorption law)

So, $e_1 \leq (e_1 \vee e_2)$ and $e_2 \leq (e_1 \vee e_2)$

Now, let $e_1 \leq y$ and $e_2 \leq y$ for some $y \in$ L

Then $e_1 = e_1 \wedge y$ and $e_2 = e_2 \wedge y$

So, $e_1 \vee y = (e_1 \wedge y) \vee y = y$ and $e_2 \vee y = (e_2 \vee y) \vee y = y$

Hence $(e_1 \vee e_2) \wedge y = (e_1 \vee e_2) \wedge (e_1 \vee y)$

$= (e_1 \vee e_2) \wedge (e_1 \vee (e_2 \vee y))$

$= (e_1 \vee e_2) \wedge ((e_1 \vee e_2) \vee y)$

$= e_1 \vee e_2$ (absorption law)

This implies $e_1 \vee e_2 \leq y$ and so $e_1 \vee e_2 = Sup\{e_1, e_2\}$

Therefore $Sup\{e_1, e_2\}$ exist in L and so $L^p = (L;\leq)$ is a lattice.

(iii) Is trivial from the proof of (ii) as the *infimum* and *suprimum* of a and b in original poset and the final poset are both equal to $e_1 \wedge e_2$ and $e_1 \vee e_2$ respectively. Therefore the partial ordering relation L and $(L^a)^p$ are identical.

So, $(L^a)^p = L$ ■

Problem 1.1.15. Prove that the absorption laws imply idempotency laws.

Solution:

By the definition of absorption law we get

$$x \wedge (x \vee y) = x \qquad \text{(i)}$$

and $$x \vee (x \wedge y) = x \qquad \text{(ii)}$$

Take $y = x \wedge y$ from (i), we get

(i) $\Longrightarrow x \wedge (x \vee (x \wedge y)) = x$

$\Longrightarrow x \wedge x = x$ which is idempotent law.

Therefore, absorption laws imply idempotent law.

Theorem 1.1.16. Let (L ; Φ) be a poset, then (L ; $\overline{\Phi}$) also a poset.

Proof:

Since Φ is reflexive, so $x \, \Phi \, x$, $\forall \, x \in L$.

This implies $x \, \overline{\Phi} \, x$, $\forall \, x \in L$.

i.e $\overline{\Phi}$ is reflexive.

Let $x \, \overline{\Phi} \, y$ and $y \, \overline{\Phi} \, x$.

Then $y \, \Phi \, x$ and $x \, \Phi \, y$, this imply $x=y$ as Φ is anti-symmetric.

Therefore $\overline{\Phi}$ is anti-symmetric.

Suppose $x \, \overline{\Phi} \, y$, $y \, \overline{\Phi} \, z$

Then $y \, \Phi \, x$ and $x \, \Phi \, y$

Thus $x \, \Phi \, y$ and $y \, \Phi \, z$ this implies $z \, \Phi \, x$ as Φ is transitive.

Therefire $x \, \overline{\Phi} \, z$ and $\overline{\Phi}$ is transitive. Hence (L ; $\overline{\Phi}$) is a poset. ∎

Definition 1.1.17.

Complete lattice: A Lattice L is called complete if $\wedge$ H and $\vee$ H exist for any subset H $\leq$ L. The concept is self-dual and half of the hypothesis is redundant.

Definition 1.1.18.

Bounded lattice: A complemented lattice is a bounded lattice in which every element has a complement.

Theorem 1.1.19. Union of two sublattices may not be a sublattice.

Proof:

Consider the lattice $L=\{1,2,3,4,6,8,12,24\}$ of factors of 24 under divisibility.

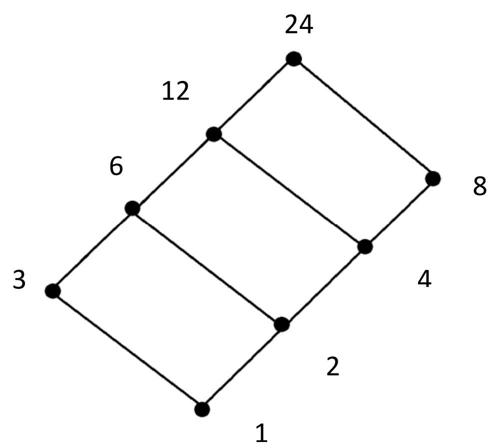

Fig. 1.3

Then $S=\{1,2\}$ and $T=\{1,3\}$ are sublattice of L. But $S \vee T=\{1,2,3\}$ is not a sublattice

as $2,3 \in S \vee T$ but $2 \vee 3=6 \notin S \vee T$. ■

Theorem 1.1.20. A lattice L is a chain iff every non empty subset of it is a sublattice.

Proof:

Let S be a non empty subset of a chain L then $a,b \in S$ implies that $a,b \in L$ implies that a,b comparable, let $a \leq b$ then $a \wedge b = a \in S$, $a \vee b = b \in S$, therefore S is a sublattice.

Conversely, let L be a lattice such that every nonempty subset of L is a sublattice. We show that L is a chain. Let $a,b \in L$ be any elements, then $\{a,b\}$ being a non empty subset of L will be a saublattice of L. Thus by definition of sublattice $a \wedge b \in \{a,b\}$ implies that $a \leq b$ or $b \leq a$ .i.e. a,b are comparable. Hence L is a chain. ■

Theorem 1.1.21. The algebra $<L; \wedge, \vee>$ is a lattice iff $< L; \wedge>$ and $< L; \vee>$ semi-lattices and $a=a \wedge b$ is equivalent to $b=a \vee b$.

Proof:

Let ∧ and ∨ are two binary relations on L. Since $< L; \wedge, \vee>$ is a lattice then ∧ and ∨ satisfy the following conditions : For all $a,b,c \in L$, $a \wedge a = a$, $a \vee a = a$; $a \wedge b=b \wedge a$, $a \vee b=b \vee a$;

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$; so $< L; \wedge>$ and $< L; \vee>$ are semi-lattices.

Let $a=a \wedge b$ then $a \vee b=(a \vee b) \vee b =b$. Conversely, let $< L; \wedge>$ and $< L; \wedge>$ are semi-lattices then the above three conditions hold. So we need only to show the absorption identities hold in L. $a \wedge (a \vee b) =a \wedge b=a$ and $a \vee (a \wedge b) =a \vee a=a$, so $< L; \wedge, \vee>$ is a lattice. ■

1.2 Some algebraic concepts :

Definition 1.2.1. A non empty set L together with two binary composition $\wedge$ and $\vee$ is said to form a lattice if $\forall a,b,c \in L$, the following conditions holds-

L_1 : Idempotency : $a \wedge a = a,\ a \vee a = a$

L_2 : Commutativity : $a \wedge b = b \wedge a,\ a \vee b = b \vee a$

L_3 : Associativity : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

$a \vee (b \vee c) = (a \vee b) \vee c$

L_4 : Absorption : $a \wedge (a \vee b) = a\ ,\ a \vee (a \wedge b) = a$

Definition 1.2.2. A poset P satisfies the descending chain condition if every non-empty subset of p has a minimal element.

Definition 1.2.3. Sublattice: A non-empty subset S of a lattice L is called a sublattice of L,

if $a, b \in S$

$\Longrightarrow a \wedge b, a \vee b \in S$.

Example 1.2.4. Let L be a lattice $L=\{1,2,3,4,6,12\}$ and S be a sublattice of L. $S=\{1,2,3,6\}$

Definition 1.2.5. Complements: Let x, y be any elements of a lattice L. If $x \wedge y = 0$ and $x \vee y = 1$ then we say y is complements of x.

Definition 1.2.6. Relative Complements: Let $[a, b]$ be an interval in a lattice L. Let $x \in [a, b]$ be any element if $\exists y \in L$ such that $x \wedge y = a, x \vee y = b$

We say that y is a complement of x relative to $[a, b]$ or y is complements of x in $[a, b]$.

Definition 1.2.7. Complemented: If every element x of an interval $[a, b]$ has at least one complement relative to $[a, b]$, the interval $[a, b]$ is said to be complemented.

If every interval in a lattice is complemented, the lattice is said to be relatively complemented.

Definition 1.2.8. Let P_1 and P_2 be two posets. A map $\Phi: P_1 \longrightarrow P_2$ is called an isotone if for $a,b \in P_1$ with $a \leq b \Longrightarrow \Phi(a) \leq \Phi(b)$ in P_2 .

Definition 1.2.9. Suppose $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ are two lattices. A map Φ: $L_1 \longrightarrow L_2$ is called a meet homomorphism if for $a,b \in L_1$, $\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)$ in L_2. On the other hand Φ: $L_1 \longrightarrow L_2$ is called a join homomorphism if for $a, b \in L_1$, $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$ in L_2.

Definition 1.2.10. Suppose $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ are two lattices. A map $\Phi: L_1 \longrightarrow L_2$ is called homomorphism if $\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)$ and $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$ in L_2, for any $a, b \in L_1$.

1.3 Ideal Lattices

Definition 1.3.1. A non empty subset I of a lattice L is called an ideal of L is

(i) $a,b \in I \Rightarrow a \vee b \in I$

(ii) $a \in I, l \in L \Rightarrow a \wedge l \in I$

If L is bounded then $\{0\}$ is always an ideal of L and is called the zero ideal.

Definition 1.3.2.

Dual Ideal: A non empty subset F of a lattice L is called a dual ideal (or filter) of L iff

(i) $x,y \in F \Rightarrow x \wedge y \in F$

(ii) $x \in F, l \in L \Rightarrow x \vee l \in F$

Definition 1.3.3.

Principal n-ideal: Let L be a lattice and $a \in L$ be any element. Let $(a]=\{ x \in L / x \leq a \}$, then $(a]$ forms an ideal of L. It is called principal ideal generated by a.

Prime Ideal: An ideal A of a lattice L is called a prime ideal of L if A is properly contained in L and whenever $a \wedge b \in A$ then $a \in A$ or $b \in A$.

Theorem 1.3.4. Intersection of two ideals is an ideal.

Proof:

Let I_1 and I_2 are two ideals of a lattice L. Since I_1, I_2 are non empty, there exists some $a \in I_1$, $b \in I_2$. Now $a \in I_1$, $b \in I_2 \subseteq L$ implies that $a \wedge b \in I_1$. Similarly $a \wedge b \in I_2$. Thus $I_1 \cap I_2 \neq \Phi$. Let $x,y \in I_1 \cap I_2$ be any elements implies that $x,y \in I_1$ and $x,y \in I_2$ implies that $x \vee y \in I_1$ and $x \vee y \in I_2$ as I_1, I_2 are ideals. So, $x \vee y \in I_1 \cap I_2$. Again if $x \in I_1 \cap I_2$ and $l \in L$ be any elements then $x \in I_1$, $x \in I_2$, $l \in L$ implies that $x \wedge l \in I_1$ and $x \wedge l \in I_2$ implies that $x \wedge l \in I_1 \cap I_2$. Hence $I_1 \cap I_2$ is an ideal. ■

Theorem 1.3.5. A non empty subset I of a lattice L is an ideal iff

(i) $a,b \in I$ implies that $a \vee b \in I$

(ii) $a \in I$, $x \leq a$ implies that $x \in I$

Proof:

Let I be an ideal of a lattice L. By definition of ideal (i) is satisfied. Let $a \in I$, $x \leq a$ then $x = a \wedge x \in I$.

Conversely, we need show that $a \in I$, $l \in L$ implies that $a \wedge l \in L$ since $a \wedge l \leq a$ and $a \in I$. By given condition $a \wedge l \in I$. Hence I is an ideal. ■

Theorem 1.3.6. Every ideal of a lattice L is prime iff L is chain.

Proof:

Let $a,b \in L$, so $a \wedge b \in L$. Consider $(a \wedge b]$ by hypothesis $I=(a \wedge b]$ is prime implies that either $a= a \wedge b$ or $b= a \wedge b$ implies that either $a \leq b$ *or* $b \leq a$. Hence L is chain.

Conversely, Let L be a chain and I be an ideal of L. Suppose, $a \wedge b \in P$, since L is chain, either $a \leq b$ or $b \leq a$ implies that $a \in I$ or $b \in I$, therefore I is prime. ■

Theorem 1.3.7. Let L be a lattice the following conditions are equivalent:

(i) L is distributive.

(ii) For any ideal I and any filter F of L,

Such that $I \cap F = \Phi$, there exists a prime ideal $P \supseteq I$ and disjoint from F. ■

CHAPTER II

Modular and distributive lattice

Introduction : Distributive lattices, modular lattices and Boolean algebra has been studied by several authors including Katrinak [1], H. Lakser [3], A.S.A Noor & M.A Latif [7], W.H Cornish [8], A Davey [9], G. Gratzer [10] and Vijjay K Khanna [11]. In this chapter we discuss distributive lattices and modular lattices which are basic concept of this thesis.

Definition 2.1.1. A lattice L is called a modular lattice if $\forall\, x,y,z \in L$, with $x \geq y$

$$x \wedge (y \vee z)=(x \wedge y) \vee (x \wedge z)=y \vee (x \wedge z)$$

Remark 2.1.2.

(i) If in the above definition $a = b$, we find $a \wedge (b \vee c) = a \wedge (a \vee c) =a$

$$b \vee (a \wedge c) = a \vee (a \wedge c) =a$$

i.e the postulate is automatically satisfied.

(ii) If $c \geq b$

Then $a \geq b, c \geq b$

$\Longrightarrow\ a \vee c \geq b, a \wedge c \geq b$

Thus $a \wedge (b \vee c)= a \wedge c$

$b \vee (a \wedge c)= a \wedge c$

(iii) Dual of the modularity postulate will real as for $a,b,c \in L$ with $a \leq b$

$a \vee (b \wedge c)= b \wedge (a \vee c)$

which is nothing but the original postulate. Hence dual of a modular lattice is modular.

Example 2.1.3. The lattices given by the following diagrams are modular.

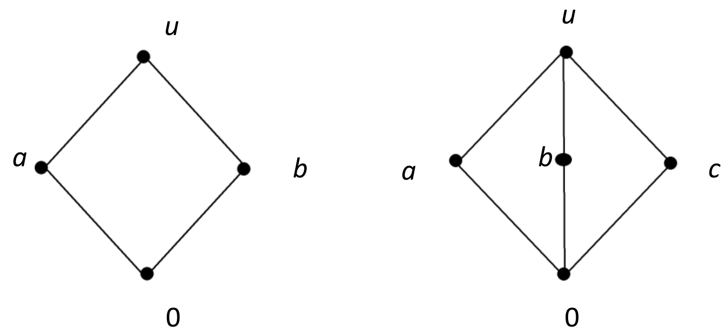

Fig. 2.1

In the first we cannot find any triplet a,b,c such that $a>b$ and c is not comparable with a or b. Hence by the remark above it is modular. By similar argument the second lattice is also seen to be modular.

Example 2.1.4. The pentagonal lattice is not modular.

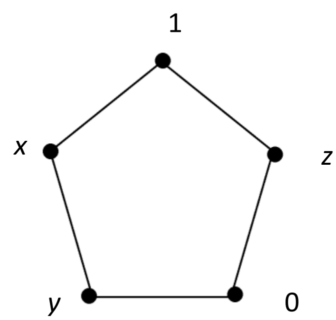

Fig. 2.2

Here, $x \wedge (y \vee z) = x \wedge 1 = x$

$(x \wedge y) \vee z = x \vee z = 1$

$x \wedge (y \vee z) \neq (x \wedge y) \vee z$

Hence the pentagonal lattice is not modular.

Lemma 2.1.5. The following inequalities hold in any lattice:

(i) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$

(ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

(iii) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

Proof:

(i) In any lattice $(x \wedge y) \leq x$, $(x \wedge y) \leq y$, $y \leq y \vee z$ implies that $(x \wedge y) \leq x$, $x \wedge y \leq y \vee z$ implies that $x \wedge y$ is a lower bound of $\{x, y \vee z\}$: $x \wedge y \leq x \wedge (y \vee z)$ (i)

Again in any lattice, $(x \wedge z) \leq x$, $(x \wedge z) \leq z$, $z \leq y \vee z$ implies that $(x \wedge z) \leq x$, $x \wedge z \leq y \vee z$ implies that $x \wedge z$ is a lower bound of $\{x, y \vee z\}$. $\therefore x \wedge z \leq x \wedge (y \vee z)$ (ii)

From (i) and (ii), we can say that $x \wedge (y \vee z)$ is upper bound of $\{x \wedge y, x \wedge z\}$.

Therefore $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$.

(ii) In any lattice, $x \leq x \vee y$, $y \leq x \vee y$, $y \wedge z \leq y$ implies that $x \vee y \geq x$, $x \vee y \geq y$, $y \geq y \wedge z$ implies that $x \vee y \geq x$, $x \vee y \geq y \wedge z$ implies that $x \vee y$ is upper bound of $\{x, y \wedge z\}$ $\therefore x \vee y \geq x \vee (y \wedge z)$ implies that $x \vee (y \wedge z) \leq x \vee y$ (i)

Again, $x \leq x \vee z$, $z \leq x \vee z$, $y \wedge z \leq z$ implies that $x \vee z \geq x$, $x \vee z \geq z$, $z \geq y \wedge z$ implies that $x \vee z \geq x$, $x \vee z \geq y \wedge z$ implies that $x \vee z$ is upper bound of $\{x, y \wedge z\}$ (ii)

From (i) and (ii)

$x \vee (y \wedge z)$ is a lower bound of $\{x \vee y, x \vee z\}$.
Therefore $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

(iii) In any lattice, $x \wedge y \leq x$, $x \leq x \vee y$

Implies that $x \wedge y \leq x \vee y$ (i)

Again $x \wedge y \leq y$, $y \leq y \vee z$

Implies that $x \wedge y \leq y \vee z$ (ii)

Also $x \wedge y \leq x$, $x \leq z \vee x$

Implies that $x \wedge y \leq z \vee x$ (iii)

From (i),(ii),(iii) we can say that $x \wedge y$ is lower bound of $\{ x \vee y , y \vee z , z \vee x \}$

$\therefore x \wedge y \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ (A)

Again $y \wedge z \leq y$, $y \leq x \vee y$ implies that $y \wedge z \leq x \vee y$ (iv)

Also $y \wedge z \leq z$, $z \leq y \vee z$

Implies that $y \wedge z \leq y \vee z$ (v)

And $y \wedge z \leq z$, $z \leq z \vee x$

$\therefore y \wedge z \leq z \vee x$ (vi)

From (iv),(v) and (vi) we can say that y ∧ z is lower bound of $\{ x \vee y , y \vee z , z \vee x \}$.

$\therefore y \wedge z \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ (B)

Similarly, $z \wedge x \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ (C)

From (A),(B) and (C) we can say that $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ is upper

bound of $\{ x \wedge y , y \wedge z , z \wedge x \}$.

$\therefore (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \leq (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ ■

Theorem 2.1.6. Dual of a modular lattice is modular.

Proof:

Let L be a modular lattice. Let $a,b,c \in L$, since L is modular $a \geq b$.

$\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = b \vee (a \wedge c) \quad \forall\, a, b, c \in L$

Now we have to show that dual of L is modular

i.e $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \forall\, a, b, c \in D$

Here D is the dual of L. Let $a, b, c \in D$ be any there element,

then $(a \wedge b) \vee (a \wedge c) = [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c]$

$= a \wedge b[(a \wedge b) \vee c]$

$= a \wedge [(c \vee a) \wedge (c \vee b)]$

$= a \wedge (b \vee c)$

Therefore D is modular. Hence dual of a modularlattice is modular. ■

Theorem 2.1.7. If L_1 and L_2 are modular iff they are Cartesian products are modular.

Proof:

Let L_1 and L_2 be modular. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in L_1 \times L_2$ be three elements with $(x_1, y_1) \geq (x_3, y_3)$. Then $x_1, x_2, x_3 \in L_1$, $x_1 \geq x_3$

$y_1, y_2, y_3 \in L_2$, $y_1 \geq y_3$ and since L_1 and L_2 are modular.

We get $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee x_3$,

$y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee y_3$

Thus $(x_1, y_1) \wedge [(x_2, y_2) \vee (x_3, y_3)]$

$= (x_1, y_1) \wedge [x_2 \vee x_3, y_2 \vee y_3]$

$= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$

$= ((x_1 \wedge x_2) \vee x_3, (y_1 \wedge y_2) \vee y_3)$

$= ((x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_3, y_3))$

$= [(x_1, y_1) \wedge (x_2, y_2)] \vee (x_3, y_3)]$

Hence $L_1 \times L_2$ is modular.

Conversely, Let $L_1 \times L_2$ be modular, let $x_1, x_2, x_3 \in L_1$, $x_1 \geq x_3$

And $y_1, y_2, y_3 \in L_2$, $y_1 \geq y_3$ then $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in L_1 \times L_2$

And $(x_1, y_1) \geq (x_3, y_3)$. Since $L_1 \times L_2$ is modular .

We find ,

$(x_1, y_1) \wedge [(x_2, y_2) \vee (x_3, y_3)] = [(x_1, y_1) \wedge (x_2, y_2)] \vee (x_3, y_3)]$

Or , $(x_1, y_1) \wedge [x_2 \vee x_3, y_2 \vee y_3] = ((x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_3, y_3))$

Or, $(x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3)) = ((x_1 \wedge x_2) \vee x_3, (y_1 \wedge y_2) \vee y_3)$

$x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee x_3$

$y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee y_3$

$\therefore$ L_1 and L_2 are modular . ■

Problem 2.1.8. Show that a lattice L is modular iff satisfies the identity

$$(x \vee (y \wedge z)) \wedge (y \vee z) = (x \wedge (y \vee z)) \vee (y \wedge z)$$

Solution:

Let $x, y, z \in L$

Firstly, $(x \vee (y \wedge z)) \wedge (y \vee z)$

$= (x \wedge (y \vee z)) \vee ((y \wedge z) \wedge (y \vee z))$ [by definition of modularity]

$= (x \wedge (y \vee z)) \vee (y \wedge z)$ $[\because (y \wedge z) \wedge (y \vee z) = (y \wedge z)]$

$\Rightarrow (x \vee (y \wedge z)) \wedge (y \vee z) = (x \wedge (y \vee z)) \vee (y \wedge z)$

Conversely , $(x \wedge (y \vee z)) \vee (y \wedge z)$

$= (x \vee (y \wedge z) \wedge ((y \vee z) \vee (y \wedge z))$

$= (x \vee (y \wedge z) \wedge (y \vee z)$ $[\because (y \wedge z) \vee (y \vee z) = (y \vee z)]$

$\therefore (x \wedge (y \vee z)) \vee (y \wedge z) = (x \vee (y \wedge z)) \wedge (y \vee z)$

Theorem 2.1.9. If a, b are any elements of a modular lattice then

$$[a_1 \wedge a_2, a_1] \cong [a_2, a_1 \vee a_2]$$

Proof:

We know an interval in a lattice is a sublattice. We establish the isomorphism.

Define a map : $[a_1 \wedge a_2, a_1] \rightarrow [a_2, a_1 \vee a_2]$ such that $\Psi(x) = x \vee a_2$.

$x \subset [a_1 \wedge a_2, a_1]$. Then Ψ is well defined as $x \in [a_1 \wedge a_2, a_1]$ implies that

$a_1 \wedge a_2 \leq x \leq a_1$ implies that $(a_1 \wedge a_2) \vee a_2 \leq x \vee a_2 \leq a_1 \vee a_2$ implies that

$a_2 \leq x \vee a_2 \leq a_1 \vee a_2$

Implies that $x \vee a_2 \in [b, a_1 \vee a_2]$. Also $x_1 = x_2$

Implies that $x_1 \vee a_2 = x_2 \vee a_2$

Implies that $\Psi(x_1) = \Psi(x_2)$,

Ψ is one **--** one as let $\Psi(x_1) = \Psi(x_2)$, then $x_1 \vee a_2 = x_2 \vee a_2$

Implies that $a_1 \wedge (x_1 \vee a_2) = a_1 \wedge (x_2 \vee a_2)$

Implies that $x_1 \vee (a_1 \wedge a_2) = x_2 \vee (a_1 \wedge a_2)$ implies that $x_1 = x_2$.

Ψ is onto as let $y \in [a_2, a_1 \vee a_2]$ be any element.

We show that $a_1 \wedge y$ is the required pre-image.

$y \in [a_2, a_1 \vee a_2]$ implies that $a_2 \leq y \leq a_1 \vee a_2$

Implies that $a_1 \wedge a_2 \leq a_1 \wedge y \leq a_1 \wedge (a_1 \vee a_2)$ implies that $a_1 \wedge a_2 \leq a_1 \wedge y \leq a_1$

Implies that $a_1 \wedge y \in [a_1 \wedge a_2, a_1]$.

Also, $\Psi(a_1 \wedge y) = (a_1 \wedge y) \vee a_2$, so we need show $y = (a_1 \wedge y) \vee a_2$

Now, $y \leq a_1 \vee a_2$ implies that $y \wedge (a_1 \vee a_2) = y$

Implies that $y = y \wedge (a_2 \vee a_1) = a_2 \vee (y \wedge a_1)$.

Hence Ψ is onto.

Again, $x_1 \leq x_2$ implies that $x_1 \vee a_2 \leq x_2 \vee a_2$

Implies that $\Psi(x_1) \leq \Psi(x_2)$

Now, $x_1 \vee a_2 \leq x_2 \vee a_2$

Implies that $a_1 \wedge (x_1 \vee a_2) \leq a_1 \wedge (x_2 \vee a_2)$

Implies that $x_1 \vee (a_1 \wedge a_2) \leq x_2 \vee (a_1 \wedge a_2)$

Implies that $x_1 \leq x_2$. Thus $x_1 \leq x_2$

Implies that $\Psi(x_1) \leq \Psi(x_2)$.

Hence Ψ is an isomorphism. ■

Modular and distributive lattice are so closely related to each other that some of the results patterning to these could be studied together.

2.2 Distributive Lattice :

Definition 2.2.1. A lattice L is called a distributive lattice if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \forall\, a, b, c \in L$$

Theorem 2.2.2. A distributive lattice is always modular but converse is not true.

Proof:

Suppose L is distributive, let $a, b, c \in L$ with $c \leq a$, then

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = a \vee (b \wedge c)$, thus L is modular .

for this converse, consider the lattice

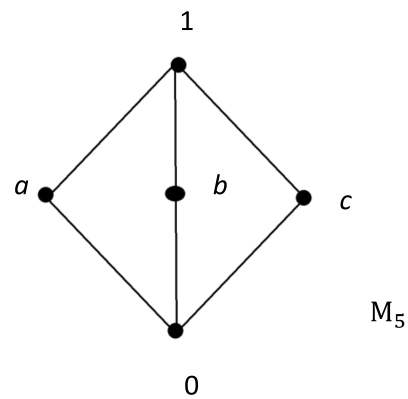

Fig. 2.3

it is easy to check that M_5 is modular. In M_5, $a \wedge (b \vee c) = a \wedge 1 = a$.

$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$. i.e., $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

Therefore L is not distributive. ■

Theorem 2.2.3. Two lattices L_1 and L_2 are distributive iff $L_1 \times L_2$ is distributive.

Proof:

Let L_1 and L_2 are distributive, let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be any three elements of $L_1 \times L_2$ then $x_1, x_2, x_3 \in L_1$ and $y_1, y_2, y_3 \in L_2$.

Now, $(x_1, y_1) \wedge [(x_2, y_2) \vee (x_3, y_3)] = (x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3)$

$= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$

$= ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3))$

$= [(x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3)]$

$= [(x_1, y_1) \wedge (x_2, y_2)] \vee [(x_1, y_1) \wedge (x_3, y_3)]$

Shows $L_1 \times L_2$ is distributive.

Conversely, let $L_1 \times L_2$ be distributive.

Let $x_1, x_2, x_3 \in L_1$ and $y_1, y_2, y_3 \in L_2$ be any elements , then

$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in L_1 \times L_2$ and as $L_1 \times L_2$ is distributive .

$(x_1, y_1) \wedge [(x_2, y_2) \vee (x_3, y_3)]$

$= [(x_1, y_1) \wedge (x_2, y_2)] \vee [(x_1, y_1) \wedge (x_3, y_3)]$

i.e, $(x_1, y_1) \wedge (x_2 \vee x_3, y_2 \vee y_3) = (x_1 \wedge x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_3, y_1 \wedge y_3)$

Or, $((x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$

$= ((x_1 \wedge x_2) \vee (x_1 \wedge x_3), (y_1 \wedge y_2) \vee (y_1 \wedge y_3))$

Which gives, $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$

$y_1 \wedge (y_2 \vee y_3) = (y_1 \wedge y_2) \vee (y_1 \wedge y_3)$

Implies that L_1 and L_2 are distributive . ■

Definition 2.2.4. For a distributive lattice let J(L) denote the set of all nonzero joint-irreducible elements regarded as a poset under the partial ordering of L. For $a \in L$ set

$$r(a) = \{x | x \leq a, x \in J(L)\}$$

$$= (a] \cap J(L)$$

Theorem 2.2.5. Let L be a finite distributive lattice, then the map $\rho: x \rightarrow r(x)$ is an isomorphism between L and H (J (L)).

Proof :

Since L is finite, every element is the join of nonzero join-irreducible elements thus ,

$$X = \bigvee r(x)$$

Showing that ρ is one-to-one. Obviously, $r(x) \cap r(y) = r(x \wedge y)$ and

so $(x \wedge y)\rho = x\rho \wedge y\rho$. The formula $(x \vee y)\rho = x\rho \wedge y\rho$ is equivalent to

$$r(x \vee y) = r(x) \cup r(y)$$

To verify this formula, note that $r(x) \cup r(y) \subseteq r(x \vee y)$ is trivial.

Now let $x \in r(x \vee y)$, then $a = a \wedge (x \vee y)$

$$= (a \wedge x) \vee (a \wedge y) ;$$

Therefore $a = a \wedge x$ or $a = a \wedge y$, since a is joint-irreducible. Thus $a \in r(x)$ or

$a \in r(y)$, that is $a \in r(x) \cup r(y)$.

Finally, we have to show that if A ∈ H(J(L)), then xρ = A for some $x \in L$. Set $x = \bigvee a$. then $r(x) \supseteq A$ is obvious. Let $a \in r(x)$;

Then $a = a \wedge x = a \wedge \bigvee A = \bigvee(a \wedge b) \;\; b \in A$.

So, $a = a \wedge b$ for some $b \in A$.

Implying that $a \in A$ since A is hereditary.. ■

Theorem 2.2.6. A modular lattice is distributive iff it does not contain any sublattice isomorphic to M_5.

Proof :

Let L be modular , but non distributive , and choose $x, y, z \in L$ such that $x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$. The free modular lattice generated by x, y, z is shown in figure

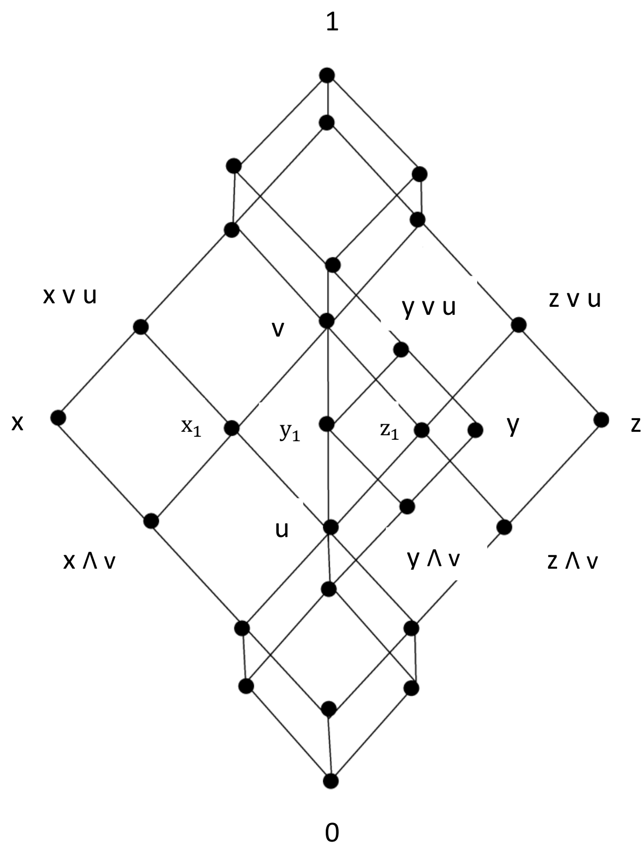

Fig 2.4

By inspecting the diagram we see that u, x_1, y_1, z_1, v from a sublattice isomorphic to M_5. Thus in any modular lattice they form a sublattice isomorphic to a quotient of M_5. But M_5 has only two quotient lattices: M_5 and the one – element lattice. In the former case we have finished the proof. In the latter case. Note that if u and v collapse, then so do $x \wedge (y \vee z)$ and $(x \wedge y) \vee (x \wedge z)$, contrary to our assumption . ■

Theorem 2.2.7. Let L be a distributive lattice, let I be an ideal. Let D be a dual of L and $I \cap D = \Phi$, then there exists a prime ideal P of L such that $P \supseteq I$.

Proof:

Let X be the set of all ideals of L containing I that are disjoint form D. Clearly X is non empty, as $I \in X$.

Let C be a chain in X and let $M = U\{ X | X \in C\}$. If $a, b \in M$, then $a \in X, b \in Y$, for some $X, Y \in C$. Since C is chain either $X \subseteq Y$ or $Y \subseteq X$.

Suppose $X \subseteq Y$, then $a, b \in Y$ since Y is an ideal $a \vee b \subset Y \subseteq M$. Also if $a \in M$ and $b \leq a$, then $a \in X$ for some $X \in C$.

Since X is an ideal, so $b \in X \subseteq M$. Therefore M is an ideal contain I. Obviously $M \cap D = \Phi$. Hence $M \in C$, so by zorn's lemma X has a maximal element, say P, we claim that P is a prime ideal. If P is not prime, then there exists $a, b \in L$ with $a,b \notin P$ such that $a \wedge b \in P$. By the maximality of $P((a] \vee P) \cap D \neq \Phi$, $((b] \vee P) \cap D \neq \Phi$. let $p \vee a \in D$ and $q \vee b \in D$ for some $p, q \in P$.

Then $x = (p \vee q) \wedge (a \vee b)$

$= (p \wedge q) \vee (a \wedge q) \vee (p \wedge b) \vee (a \wedge b) \in P$

Which implies that $x \in P \cap D$. which gives a contradiction. Therefore Φ must be a prime ideal.

■

Corollary 2.2.8. Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.

Proof :

Let $I_1 = \cap (P/P \supseteq I$, P is a prime ideal of L) if $I \neq I_1$, then there is an $a \in I_1 - I$, and (corollary 3.1.2)[L be a distributive lattice, let I be an ideal of L, and let $a \in L$ and $a \notin I$. Then there is a prime ideal P such that $P \supseteq I$ and $a \notin P$]. But then $a \notin P \supseteq I$ is a contradiction.

CHAPTER III

Prime Ideal and Minimal Prime n-Ideal

3.1 Prime ideal of a lattice:

Introduction: Prime ideal and Pseudo complemented of a lattice have been studied by several authors including [7]. In this chapter we discuss prime ideals, minimal prime ideals and minimal prime n- ideals of a lattices. In section one of this chapter we give some basic properties of prime ideals which will be needed in the next part.

In section two of this chapter we have given characterization of minimal prime ideals of a Pseudo complemented distributive lattice. Then we have show that every Pseudo complemented lattice is generalized stone.

Definition: (Dual ideal) : A non empty subset F of a lattice L is called dual ideal of L if (1) $x, y \in F$ implies that $x \wedge y \in F$

(2) $x \in F$, $d \in L$ implies that $x \vee d \in L$

Let $L = \{1,2,5,10\}$ be the lattice under divisibility. Then $\{10\},\{5,10\},\{2,10\}$ are all dual ideals of lattice L.

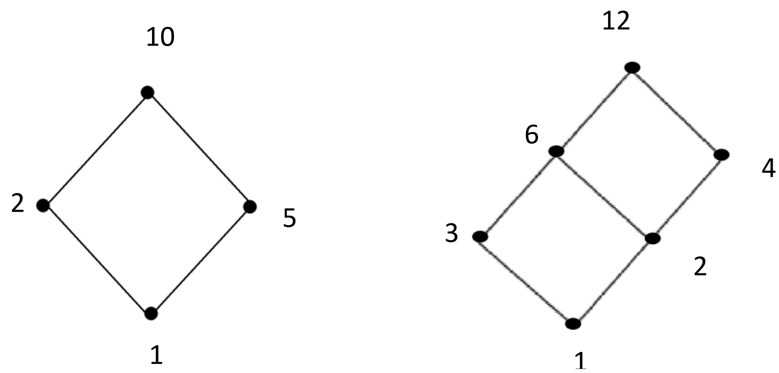

Fig 3.1

A proper ideal P of a lattice L is called a prime ideal if for any $x, y \in L$ and $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. Let $L= \{1, 2, 3, 4, 6, 12\}$ of factors 12 under divisibility forms a lattice then $\{1, 2, 4\}$ be a prime ideal of L. But in the lattice $\{1, 2, 5, 10\}$ under divisibility $\{1\}$ input a prime ideal because $2 \wedge 5=1 \in \{1\}$ But $2,5 \notin \{1\}$.

Theorem 3.1.1. Every ideal of a lattice L is prime ideal if and only if the lattice L is chain.

Proof:

Let L be chain, Let P be any proper ideal of L. If $a \wedge b \in P$ then as a, b are in a chain, they are comparable. Let $a \leq b$. then $a \wedge b = a$.

Thus $a \wedge b \in P \Longrightarrow a \in I \Longrightarrow P$ is prime. Conversely, Let every ideal in P be prime. To show that L in a chain, Let $a,b \in L$ be any elemants, Let $P = \{x \in L \,/\, x \leq a \wedge b\}$ then P in easily seen to be an ideal of L. Thus P is a prime ideal .

Now $a \wedge b \in I$, P is prime, thus $a \in P$ *or* $b \in P \Longrightarrow a \leq a \wedge b$ or
$b \leq a \wedge b \Longrightarrow a \wedge b \leq a \leq a \wedge b$ or $a \wedge b \leq b \leq a \wedge b \Longrightarrow a = a \wedge b$ *or* $b = a \wedge b \Longrightarrow a \leq b$ *or*
$b \leq a$. L is a chain. ■

Corollary 3.1.2. Let L be a distributive lattice, Let I be an ideal of L, and let $a \in I$ be an ideal of L, and let $a \in L$ and $a \in I$. Then there is a prime ideal P such that $P \supseteq I$ and $a \notin P$.

Theorem 3.1.3. Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.

Proof:

Let $I_1 = \cap\{ P \mid P \supseteq I$, P is a prime ideal of L$\}$ if $I \neq I_1$, then there is an $a \in I_1 - I$, and so by corollary 3.1.2. There in a prime ideal P, with $P \supseteq I$ and $a \notin P$. But then $a \notin P \supseteq I_1$ and is a contradiction. ■

Theorem 3.1.4. Let P be a prime ideal of a lattice L, then $L-P$ is a dual prime ideal.

Proof:

Since P is a prime ideal, therefore P is non empty, $\therefore L-P$ is a proper subset of L.

Let $x,y \in L-P$. Then $x,y \in L$, $x,y \notin P \Longrightarrow x \wedge y \in L$, $x \wedge y \notin P$(as $x \wedge y \in P \Longrightarrow x \in P$ or $y \in P$ as P is prime) $\Longrightarrow x \wedge y \in L-P$.

Again, let $x \in L - P, I \in L$. Then $x \in L, x \notin P, I \in L$
$\Longrightarrow x \vee I \in L, x \notin P, \Longrightarrow x \vee I \in L, x \vee I \notin P$ (as $x \vee I \in P \Longrightarrow x \in P$ as $x \leq x \vee I$).
Thus $x \vee I \in L-P$. i.e $L-P$ is dual ideal.

Now let $x \vee y \in L-P$, then $x \vee y \in L, x \vee y \notin P \Longrightarrow x,y \in L, x \notin P$ or $y \notin P$
(as $x,y \in P \Longrightarrow x \vee y \in P) \Longrightarrow x \in L-P \; or \; y \in L-P$.

i.e $L-P$ is a dual prime ideal. ■

3.2 Minimal prime ideals:

A prime ideal P of a lattice L is called minimal, if there does not exists a prime ideal Q such that $Q \subset P$.

The following lemma is a fundamental result in lattice theory: e . f [7,lemma 4pp ,169]. Through our proof is similar to their proof, we include the proof for the convenience of the reader.

Minimal prime ideals and stone (generalized) lattices have been studied extensively by many authors including [12], [13],[14],[15],[16],[17],[18] and [19]. Chen and in Gratzer [20] and [21] studied the construction and structures of stone lattices. Katrinak has given a new proof of construction theorem for stone algebras in [22] and studied these algebras in [23].

Theorem 3.2.1. Let L be a lattice with 0. Then every prime ideal contains a minimal prime ideal.

Proof:

Let P be a prime ideal of L and let R denote the set of all prime ideals Q contained in P. Then R is non-void, since $0 \in Q$ and Q is an ideal: infact, Q is prime. Indeed, if $a \wedge b \in Q$ for some $a, b \in L$, then $a \wedge b \in X$ for all $X \in C$; since X is prime, either $a \in X$ or $b \in X$. Thus either $Q= \cap(X|a \in X)$ or $Q = \cap (X| b \in X)$ proving, that a or $b \in Q$. Therefore, we can apply to R the dual form of Zorn's lemma to conclude the existence of a minimal member of R. ■

Theorem 3.2.2: Let L be a distributive lattice with 0, the following conditions are equivalent.

(i) L is normal.

(ii) Each prime ideal of L contains a unique minimal prime ideal.

(iii) Each prime filter of L is contained in a unique ultrafilter of L.

(iv) Any two distinct minimal prime ideals are comaximal.

(v) For all $x,y \in L$, $x \wedge y= 0$ implies $(x]^* \vee (y]^* = L$.

(vi) $(x \wedge y]^* = (x]^* \vee (y]^*$ for all $x,y \in L$. ■

Remark: Here $(x]^*$ we means relatively Pseudo complement of $(x]$.

Dense set: $D(L) = \{ a \in L : a^*= 0\}$, $D(L)$ is called the dense set.

Theorem 3.2.3. In a stone algebra every prime ideal contains exactly one minimal prime ideal.

Proof:

Let L be a Stone algebra and let P be a prime ideal of L.We need prove that P contains exactly one minimal prime ideal. Suppose P contains two distinct minimal prime ideals Q_1 and Q_2. Choose $x \in Q_1 - Q_2$ $(Q_1 \not\subset Q_2$, since Q_2 is minimal and $Q_2 = Q_1$, hence $Q_1 - Q_2 \neq \emptyset)$;

Since $x \wedge x^* = 0 \in Q_2$, $x \notin Q_2$ and Q_2 is prime, so $x^* \in Q_2$, $L - Q_1$ is maximal dual prime ideal, hence it is a maximal dual ideal of L.

Thus, $(L - Q_1) \vee [x) = L$ and so, $x \wedge a = 0$ for some $a \in L - Q_1$, therefore, $x^* \geq a \in L - Q_1$ implies that $x^* \notin Q_1$. Hence $x^* \in Q_2 - Q_1$. Similarly, $x^* \in Q_1$, so x^* and x^{**} both contained in p. implies that $1 = x^* \vee x^{**} \in P$, which is a contradiction that P is a prime ideal of L. Thus in a Stone algebra every prime ideal contains exactly one minimal prime ideal. ■

Theorem 3.2.4. Let L be a sectionally Pseudo complemented distributive lattice and p be a prime ideal in L. Then the following conditions are equivalent:

(i) p is minimal
(ii) $x \in p$ implies $(x]^* \notin p$
(iii) $x \in p$ implies $(x]^{**} \subseteq p$
(iv) $p \cap D(L) = \emptyset$

Proof :

(i) implies (ii)

Let P be minimal and (ii) fail, that is $a^* \in P$ for some $a \in P$. Let $D = (L-P) \vee [a)$, we claim that $0 \notin D$. Indeed, if $0 \in D$, then $q \wedge a = 0$ for some $q \in L-P$, which implies that $q \leq a \in P$, a contradiction. Thus there exists a prime ideal Q disjoint to D. Then $Q \subseteq P$ since $Q \cap (L-P) = \emptyset$, and $Q \neq P$ since $a \notin Q$, contradicting the minimality of P.

(ii) implies (iii)

Indeed, $x^* \wedge x^{**} = 0 \in P$ for any $x \in L$ thus if $x \in P$ then by (ii) $x^* \in P$, implying that $x^{**} \in P$.

(iii) implies (iv)

If $a \in P \cap D(L)$ for some $a \in L$, then $a^{**} = 1 \notin P$, a contradiction to (iii), thus $P \cap D(L) = \emptyset$.

(iv) implies (i)

If P is not minimal, then $Q \subset P$ for some prime ideal Q of L. Let $x \in Q - P$. Then $x \wedge x^* = 0 \in Q$ and $x \notin Q$: then $x^* \in Q \subset P$ which implies that $x \vee x^* \in P$. As $x \vee x^* \in D(L)$; thus we obtain $x \vee x^* \in P \cap D(L)$, contradicting (iv). Hence P is minimal. ■

Definition:

(stone lattice): A distributive Pseudo complemented lattice L is called a stone lattice if for each $a \in L$, $a^* \vee a^{**} = 1$.

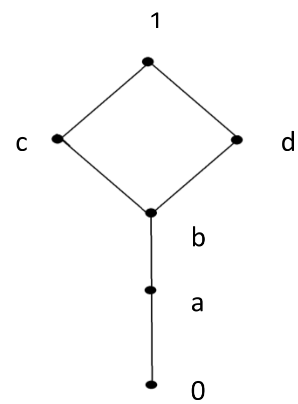

Fig. 3.3

Theorem 3.2.5. A prime ideal P of a stone algebra L is minimal iff $P = [P \cap S(L)]_L$.

Proof:

Suppose P is minimal, let $x \in [P \cap S(L)]_L$. Then $x \leq r$ for some $r \in P \cap S(L)$ implies that $r \in P$ and $r \in S(L)$ implies that $r \in P$ implies that $x \in P$ implies that

$$[P \cap S(L)]_L \subseteq P \qquad \text{(i)}$$

Again let $x \in P$, since P is minimal so, $x^{**} \in P$, Then $x \in P \cap S(L)$, as $x \leq x^{**}$.
so $x \in [P \cap S(L)]_L$ implies that $P \subseteq [P \cap S(L)]_L$ (ii)

From (i) and (ii) $P = [P \cap S(L)]_L$ Conversely, let $P = [P \cap S(L)]_L$ and let $x \in P$ then $x \leq r$ for some $r \in P \cap S(L)$
implies that $x^{**} \leq r^{**} = r$ implies that $x^{**} \in P$. Hence P is minimal. ■

Theorem 3.2.6. A distributive lattice with Pseudo complementation is a Stone algebra iff every prime ideal contains exactly one minimal prime ideal (G. Gratzer and E. T. Schmidt [1957b]).

Proof:

Let L be distributive lattice with Pseudo complementation. If L is a Stone algebra, every prime ideal contains exactly one minimal prime ideal.

Conversely, let L is not a Stone lattice and let $a \in L$ such that $a^* \vee a^{**} \neq 1$. Then there exist a prime ideal R such that $a^* \vee a^{**} \in R$. We claim that $(L\text{-}R) \vee a^*) = L$ then there exist an $x \in L\text{-}R$ such that $x \wedge a^* = 0$.

Then $a^{**} \geq x \in L-R$ implies $a^{**} \in L-R$. Which is a contradiction. So $(L\text{-}R) \vee [a^*) \neq L$. Let F be a minimal dual prime ideal containing $(L-R) \vee [a^*)$ and let G be a minimal dual prime ideal containing $(L-R) \vee [a^*)$. We set $P = L-F$ and $Q = L-G$. Then P and Q are minimal prime ideals such that $P, Q \subseteq R$. Moreover $P \neq Q$, because $a^* \in F = L-P$ and hence $a^* \in P$; thus $a^{**} \in P$ but $a^{**} \notin Q$. ∎

Theorem 3.2.7. Let L be a distributive with 0 and 1. For an ideal I of L. We set $I^* = \{x|x \wedge i = 0$ for all $i \in I\}$; Let P be a prime ideal of L. Then P is minimal prime ideal iff $x \in P$ implies that $(x]^* \subseteq P$ (T.P. Speed).

Proof:

By the definition of $I^*, (x]^* = \{y|y \wedge x = 0\}$ as $x^* \wedge x = 0$ implies that $x^* \in (x]^*$ implies that $(x^*] \subseteq (x]^*$, again let $z \in (x]^*$, then then $z \wedge x = 0$ implies that $z \leq x^*$ implies that $z \in (x^*]$ implies that $(x]^* \subseteq (x^*]$ implies that $(x]^* = (x^*]$. Now suppose P be a minimal prime ideal and $x \in P$, then by the theorem $x^* \notin P$ implies that $(x^*] \nsubseteq P$ implies that $(x]^* \subseteq P$. Conversely, if for $x \in P$, $(x]^* \nsubseteq P$ and if possible. Let P is not minimal then there exist a prime ideal Q such that $Q \subset P$. Let $x \in P \subset Q$.

Now, $x^* \wedge x = 0 \in Q$ implies that $x^* \in Q$ implies that $x \in P$ implies that $(x^*] \subseteq P$ implies that $(x]^* \subseteq P$ which is a contradiction. Hence the proof. ∎

Theorem 3.2.8. Every Boolean lattice is a Stone lattice. But the conversely is not necessary true.

Proof :

Let L be a Boolean lattice. Then for each $a \in L$, its complement a' is also the Pseudo complement of a. Moreover, $a^* \vee a^{**} = a' \vee a'' = a' \vee a = 1$. Hence L is also Stone.

Observe that 3-elements chain is a Stone lattice.

For $a^* \vee a^{**} = 0 \vee 0^* = 0 \vee 1 = 1$. But it is not Boolean, as a has no complement.

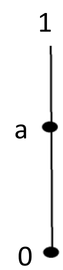

Fig. 3.4

In theorem 3.2.3, we have proved that in a Stone lattice every prime ideal contains a unique minimal prime ideal. In the following lattice, observe that $(c]$ is a prime ideal and it contains two minimal prime ideals $(a]$ and $(b]$. Hence it is not a Stone lattice.

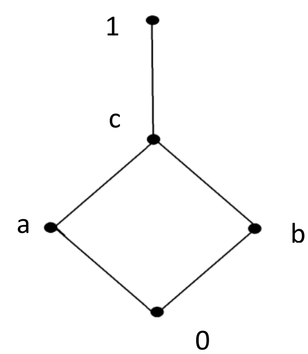

Fig. 3.5

Also by 3.1.1, we know that in a Stone lattice L, $a \vee b \in S(L)$ for all $a, b \in L$. In above lattice observe that $a \vee b = c \notin S(L)$. Hence L is not Stone.

Let L be a Stone lattice, then $S(L) = \{ a^* | a \in L \}$ is called skeleton of L. The elements of $S(L)$ are called skeletal. L is dense if $S(L) = \{0,1\}$, $< S(L) ; \wedge, \vee, *, 0, 1 >$ is a Boolean algebra. ■

Definition:

(Generalized stone lattice): A lattice L with 0 is called generalized stone lattice if $(x]^* \vee (x]^{**} = L$ for each $x \in L$.

Theorem 3.2.9. A distributive lattice L with 0 is a generalized stone lattice if and only if each interval $[0,x]$, $0 < x \in S$ is a stone lattice.

Proof :

Let L with 0 be a generalized stone and let $P \in [0,x]$. Then $(p]^* \vee (P]^{**} = L$. So $x \in (p]^* \vee (P]^{**}$ implies $x = r \vee s$ for some $r \in (p]^*$, $s \in (p]^{**}$. Now $r \in (p]^*$ implies $r \wedge P = 0$ also $0 \leq r \leq x$. Suppose $t \in [0,x]$ such that $t \wedge P = 0$, then $t \in (p]^*$ implies $t \wedge s = 0$. Therefore, $t \wedge x = t \wedge (r \vee s) = (t \wedge r) \vee (t \wedge s) = (t \wedge r) \vee 0 = (t \wedge r)$ implies $t = (t \wedge r)$ implies $t \leq r$. So r is the relative Pseudo complement of P in $[0,x]$, i.e $r=p^*$ since $s \in (p]^{**}$ and $r \in (p]^*$, So $s \wedge r = 0$. Let $q \in [0,x]$. Such that $q \wedge r = 0$. Then as $x = r \vee s$, so $q \wedge x = (q \wedge r) \vee (q \wedge s)$ implies $q = q \wedge s$ implies $q \leq s$. Hence, s is the relative Pseudo complement of $r = p^*$ in $[0,x]$ i.e $s = p^{**}$ implies $x = r \vee s = p^* \vee p^{**}$. Thus $[0,x]$ is a stone lattice.

Conversely, suppose $[0,x]$. $0 < x \in L$ is a stone lattice. Let $p \in L$, then $p \wedge x \in [0,p]$. Since $[0,p]$ is a stone lattice, then $(p \wedge x)^* \vee (p \wedge x)^{**} = p$, where $(p \wedge x)^*$ is the relative Pseudo complement of $(p \wedge x)$ in $[0,p]$.

Therefore $p \in ((p] \cap (p \wedge x]^* \vee ((p] \cap (p \wedge x]^{**}$, So, we can take $p = r \vee s$, for $r \in (p \wedge x]^*$, $s \in (p \wedge x]^{**}$. Now, $r \in (p \wedge x]^*$ implies $r \wedge p \wedge x = 0$ implies $r \wedge x = 0$ implies $r \in (x]^*$ and $s \in (p \wedge x]^{**}$. Now $p \wedge x \leq x$ implies $(p \wedge x]^{**} \subseteq (x]^{**}$, and so $s \in (x]^{**}$. Therefore $p = r \vee s \in (x]^* \vee (x]^{**}$ and so, $L \subseteq (x]^* \vee (x]^{**}$. But $(x]^* \vee (x]^{**} \subseteq L$ is obvious.

Hence $(x]^* \vee (x]^{**} = L$ and so L is generalized stone. ∎

Theorem 3.2.10. A distributive Pseudo complemented lattice is a Stone lattice L if and only if for any two minimal prime ideals P and Q. $P \vee Q = L$.

Proof:

Suppose L is a Stone lattice and P, Q are two minimal prime ideals. If $P \vee Q \neq L$ then there exists a prime ideal R containing $P \vee Q$. This means that R contains two minimal prime ideals, which is a contradiction to theorem 3.2.6 as L is a Stone, therefore $P \vee Q = L$.

Conversely, suppose the given condition holds and R is a prime ideal of L. Then R cannot contain two minimal prime ideals P and Q, as otherwise $R \supset P \vee Q = L$. Therefore again by theorem 3.2.6, L is Stone. ■

3.3 Minimal prime n-ideal

Minimal prime ideals and Stone (generalized) lattices have been studied extensively by many authors including [12], [13], [14], [15]. Chen and in Gratzer [16] and [17] studied the construction and structures of Stone lattices. Katrinak has given a new proof of construction theorem for Stone algebras in [18] and studied these algebras in [19], [20] and [21].

In this part we introduce the concept of minimal prime n-ideals and generalize some of the results on minimal prime ideals. Then we used these results to generalize several important results on stone and generalized stone lattices in lattices in terms of n-ideals.

A prime n-ideal P is said to be a minimal prime n-ideal belonging to n-ideal I if ,

(i) $I \subseteq P$, and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

A prime n-ideal P of L is called a minimal prime n-ideal if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

Theorem 3.3.1. Let L be lattice with medial element n. Then every prime n ideal contains a minimal prime n-ideal.

Proof:

Let p be a prime n-ideal of L and let R be the set of all prime n-ideal Q contained in p. Then R is non-void, since $P \in R$. If C is a chain in R and $Q = \cap(x{:}x \in C)$, then Q is a non-empty as $n \in Q$ and Q is an n-ideal, in fact, Q is prime.

Indeed, if $m(a, n, b) \in Q$ for some $a, b \in L$, then $m(a, n, b) \in x$ for all $X \in C$. since X is prime, either $a \in x$ or $b \in x$. Thus, either $Q = \cap(x{:}a \in x)$ or $Q = \cap(x{:}b \in x)$, proving that $a \in Q$ or $b \in Q$. Therefore, we can apply to R the dual form of zone's lemma to conclude the existence of a minimal member of R.

If L is a distributive lattice with $n \in L$, then we already know that $F_n(L)$ is a distributive lattice with $\{n\}$ as the smallest element. So we can talk on the sectionally Pseudo complementness of $F_n(L)$ is called sectionally Pseudo complemented if each interval $[\{n\}, <a_1 \dots\dots\dots\dots a_r>n]$ is Pseudo complemented.

That is for $\{n\} \subseteq < b_1 \dots\dots\dots\dots b_r>n \subseteq <a_1 \dots\dots\dots\dots a_r>n$. relative Pseudo-complement $< b_1 \dots\dots\dots\dots b_r>n$ in $[\{n\}, <a_1 \dots\dots\dots\dots a_r>n]$ belongs to $F_n(L)$.

Now we give a characterization of minimal prime n-ideals of a distributive lattice L. when $F_n(L)$ is seasonally Pseudo complemented. To do this we establish the following theorem. ■

Theorem 3.3.2. Let L be a distributive lattice and $n \in L$ be a medial element. Then for any $i, j \in I_n(L)$, $(I \cap J)^* \cap I = J^* \cap I$.

Proof:

Since $I \cap J \subseteq J$. So $R.H.S \subseteq L.H.S$. To prove the reverse inclusion, let $x \in L.H.S$. Then $X \in I$ and $m(x, n, t) = n$ for all $t \in I \cap J$. Since $x \in I$, So $m(x, n, j) \in I \cap J$. Thus $m(x, n, m(x, n, j)) = n$. But it can be easily seen that $m(x, n, m(x, n, j)) = m(x, n, j)$. Thus implies $m(x, n, j) = n$ for all $j \in J$. Hence $x \in R.H.S$ and so $L.H.S \subseteq R.H.S$. Thus $(I \cap J)^* \cap I = J^* \cap I$.
■

Theorem 3.3.3. Suppose n is medial element of a lattice L. If $I \subsetneq J$. $I, J \in I_n(L)$ then (i) $I = I^* \cap J$ and (ii) $I^{**} = I^{**} \cap J$.

Proof :

(i) is trival. For (ii), using (i) we have, $I^{**} = (I^*)^* \cap J = (I^* \cap J)^*$. Thus, $I^{**} = I^{**} \cap J$. ■

Theorem 3.3.4. Let n be a medial element of a distributive lattice L. Suppose $F_n(L)$ is sectionally Pseudo complemented distributive lattice and p is a prime n-ideal of L. Then the following conditions are equivalent.

(i) P is minimal.

(ii) $x \in P$ implies $< x >_n^* \not\subset P$.

(iii) $x \in P$ implies $< x >_n^{**} \subseteq P$.

(iv) $P \cap D(< t >_n) = \emptyset$ for all $t \in L - P$, where $D(< t >_n) = \{ x \in L < t >_n : < x >_n^* = \{n\} \}$

Proof:

(i) $\Longrightarrow$ (ii)

suppose P is minimal. If (ii) fails, then there exists $x \in P$ such that $< x >_n^* \subseteq P$. since P is a prime n-ideal, then P is a prime ideal or a prime filter. Suppose P is a prime ideal. Let $D = (L - P) \vee [x)$. We claim that $n \notin D$. If $n \in D$, then $n = q \wedge x$ for some $q \in L - P$.

Then $< q >_n \cap < x >_n = < (q \wedge x) \vee (q \wedge n) \vee (x \wedge n) >_n = \{n\}$ implies $< q >_n \subseteq < x >_n^* \subseteq P$. Thus $q \in P$ which is contradiction. Hence $n \notin D$. Then there exist a prime n-ideal Q with $Q \cap D = \emptyset$. Then $Q \subseteq P$ as $Q \cap (L-P) = \emptyset$ and $Q \neq P$, since $x \notin Q$. But this contradicts the minimality of P.

Hence $< x >_n^* \subseteq P$. Similarly, we can prove that $< x >_n^* \leq P$ if P is a prime filter.

(ii) $\Longrightarrow$ (iii)

Suppose (ii) holds and $x \in P$. Then $< x >_n^* \not\subset P$. Since $< x >_n^* \cap < x >_n^{**} = \{n\} \subseteq P$ and P is prime, so $< x >_n^{**} \subseteq P$.

(iii) $\Longrightarrow$ (iv)

Suppose (iii) holds and $t \in L - P$. Let $x \in P \cap D(< t >_n)$, Then $x \in P$, $x \in D(< t >_n)$. Thus $< x >_n^* = \{n\}$ and so $< x >_n^{**} = < t >_n$ By (iii) $x \in P$ implies $< x >_n^{**} \subseteq P$. Also by theorem, $< x >_n^{**} \cap < x >_n^{**} \cap < t >_n$. Hence $< x >_n^{**} \cap < t >_n = < t >_n$ and so $< t >_n \subseteq < x >_n^{**} \subseteq P$. That is $t \in P$, which is a contradiction. Therefore $P \cap D(< t >_n) = \emptyset$ for all $t \in L - P$.

(iv) $\Longrightarrow$ (i)

Suppose P is not minimal. Then there exists a Prime n-ideal $q \subset P$ Let $x \in P - Q$. Since $< x >_n \cap < x >_n = \{n\} \subseteq Q$ so $< x >_n^* \subseteq Q \subset P$. Thus $< x >_n \vee < x >_n^* \subseteq P$.

Choose any $t \in L - P$, then $< t >_n \cap (< x >_n \vee < x >_n^* \subseteq P)$ Now $< t >_n \cap (< x >_n \vee < x >_n^*) = (< t >_n \cap < x >_n) \vee (< t >_n \cap < x >_n^*) = < m(t,n,x) >_n \vee (< t >_n \cap < x >_n)^* < t >_n = < m(t,n,x) >_n \vee (< m(t,n,x) >_n^* \cap < t >_n) = < m(t,n,x) >_n \vee < m(t,n,x) >_n^*$ where $< m(t,n,x) >_n^*$ is the relative Pseudo complement of $< m(t,n,x) >_n$ in $< t >_n$. Since $F_n(L)$ is sectionally Pseudo complemented $< m(t,n,x) >_n^*$ is finitely generated and so $< m(t,n,x) >_n \vee < m(t,n,x) >_n^*$ is a finitely generated n-ideal contained in $< t >_n$.

Therefore $< m(t,n,x) >_n \vee < m(t,n,x) >_n{}^* = < r >_n$ for some $r \in < t >_n$ Moreover, $< r >_n{}^* = < m(t,n,x) >_n \cap < m(t,n,x) >_n{}^{**} = \{n\}$. Thus, $r \in P \cap D(< t >_n)$. Which is a contradiction. Therefore, P must be minimal. ■

CHAPTER IV

Distributive Lattice with Pseudo complementation

Introduction: In lattice theory there are different classes of lattices known as variety of lattices class of Boolean lattice is of course the most powerful variety. Throughout this chapter we will be concerned with another large variety known as the class of distributive Pseudo complemented lattice. Pseudo complemented lattice have been studied by several authors. [2],[3],[4],[8],[5],[6]

A Pseudo complemented distributive lattice is a distributive lattice L with 0, 1 such that for each $a \in L$ there is a greatest element a^* which is disjoint with a. The problem referred to above is then: what is the most general Pseudo complemented distributive lattice in which $a^* \vee a^{**} = 1$.

We shall deal exclusively with Pseudo complemented distributive lattices. There are two concepts that we should be able to distinguish: a lattice $<L; \wedge, \vee>$ in which every element has a Pseudo complement, and an algebra $<L; \wedge, \vee, 0, 1>$ where $<L; \wedge, \vee, 0, 1>$ is a bounded lattice and where for every $a \in L$. The element a is a Pseudo complement of a. We shall call the former a Pseudo complemented lattice and the latter a lattice with Pseudo complementation (as an operation) the same kind of distinction that we make between Boolean lattices and Boolean algebras. Thus a Pseudo complementation is an algebra $< 2,2 >$, where as a lattice with Pseudo complementation is an algebra of type $< 2,2,1,0,0 >$. To see the difference in viewpoint, consider the finite distributive lattice of the following figure. As a distributive lattice has twenty-five sublattices and eight congruences, as a lattice with Pseudo complementation has three sub algebras and five congruences.

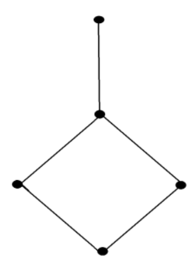

Fig. 4.1

Thus , for a lattice with Pseudo complementation L, a subalgebra L_1 is a $\{0,1\}$ sublattice of L closed under $*$ (i.e $a \in L_1$ implies that $a^* \in L_1$). A homomorphism Φ is a $\{0,1\}$ homomorphism that also satisfies $(x,\Phi)^* = x^*\Phi$. If there is any danger of confusion, we call such a homomorphism a-homomorphism. similarly, a congruence relation Θ will have the substitution property also for $* : a \equiv b\ \Theta$ implies that $a^* \equiv b^*\ (\Theta)$.

Definition :

Pseudo complemented element : Let L be a lattice with 0 and 1 for an element $x \in L$, element $x^* \in L$ is called Pseudo complement of x if $x \wedge x^* = 0$ and $x \wedge y = 0$ ($\forall y \in L$) implies $y \leq x^*$.

Definition :

Pseudo complemented lattice : Let L be a bounded distributive lattice, let $a \in L$, an element $a^* \in L$ is called a Pseudo complemented of a in L if the following conditions hold i) $a \wedge a^* = 0$ (ii) $\forall x \in L$, $a \wedge x = 0$ implies that $x \leq a^*$.

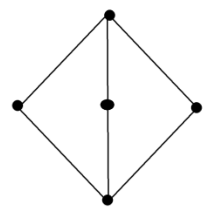

Fig 4.2

Also bounded lattice L is called Pseudo complemented if its every element has a Pseudo complement.

For a lattice L with 0 we can talk about sectionally Pseudo complemented lattice.

A lattice L with 0 is called sectionally Pseudo complemented if interval $[0,x]$ for each $x \in L$ is Pseudo complemented of course every finite distributive lattice is sectionally Pseudo complemented.

Example :

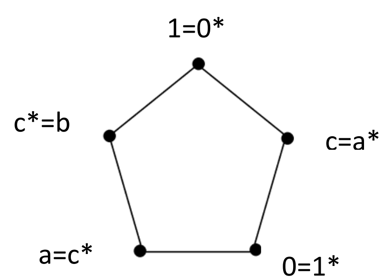

Fig. 4.3

The lattice $L = \{0,a,b,c,1\}$ shown by the fig. 4.3 is Pseudo complemented.

S(L): $S(L) = \{a^* | all\ a \in L\}$ = Set of all pseudo complemented lattice.

1. Introduction of some stone algebra : A complete lattice is called algebraic if every element is the join of compact elements.

Example : Let L be a join semi lattice with 0 then $I(L)$, the set of all ideas of L under " $\leq$ " is an algebraic lattice.

In the literature, algebraic lattice are also called compactly generated lattice. Just as for lattices a nonvoid subset I of a join semi lattices S is an ideal if, for $a,b \in S$, we have $a \vee b \in I$ iff a and $b \in I$. Again $I(S)$ is the poset of all ideals of S partially ordered under set inclusion. If S has a zero, then $I(S)$ is a lattice.

Theorem 4.1.1. A lattice L is algebraic iff it is isomorphic to the lattice of all ideals of a join semi-lattice with 0.

Proof:

Let S be a join semi lattice with 0, we have to prove that $I(S)$ is a complete lattice, we claim that $\forall\ a \in S$, $[a]$ is a compact in $I(S)$.

Let $X \subseteq I(S)$ and $(a] \subseteq \vee\ (I|\ I \in\ X)$ now $\vee(I|\ I \in\ X) =\{\ x|x \leq t_1\ \vee \dots\dots\dots \vee t_n,\ t_i \in\ I_i, I_i \in\ X\}$ Therefore $a \leq t_1\ \vee \dots\dots\dots\dots\dots\ \vee t_n,\ t_i \in\ I_i, I_i \in\ X$ Thus with $X_1 = \{\ I_1 \dots\dots\dots\dots I_n\ \}(a] \leq\ \vee(I_i \in\ X_i \subseteq X)$ Therefore $(a]$ is compact in $I(S)$. Now for any $I \in I(S)$, $I =\ \vee((a]\ |\ a \in\ L)$ Hence $I(S)$ is algebraic and so any lattice L is isomorphic to $I(S)$ is also algebraic.

Conversely, let L be an algebraic lattice and let S be the set of all compact elements of L obviously $0 \in S$.

Moreover clearly join of two compact elements is again a compact element. So S is a join semi lattice with 0. Now consider the map $\Phi:L\rightarrow I(L)$ is define by $\Phi\ (a) = \{x \in\ S\ |x \leq a\}$ obviously Φ maps L into $I(S)$. By the definition of an algebraic lattice $a =\ \vee\Phi\ (a)$ and so Φ is one-one.

To prove that Φ is onto, let $I \in I(S)$, $a =\ \vee I$ then $\Phi(a) \supseteq I$. Now, let $x \in\ \Phi(a)$, then $x \in S$, $x \leq a\ \vee L$. By compactness of x there exists a finite subset $I_1 \subseteq I$ such that $x \leq\ \vee I_1$. This implies $x \in I$ and so $I \in\ \Phi(a)$. Therefore Φ is onto.

Also $\Phi(a \wedge b) = \{x \in\ S\ |x \leq a \wedge b\}$
$= \{x \in\ S\ |x \leq a\ \} \wedge \{x \in\ S\ |x \leq b\ \} = \Phi(a) \wedge \Phi(b)$

Also $\Phi(a \vee b) = \{x \in\ S\ |x \leq a \vee b\}$
$= \{x \in\ S\ |x \leq a\ \} \vee \{x \in\ S\ |x \leq b\ \} = \Phi(a) \vee \Phi(b)$
Φ is a homomorphism. Therefore it is an isomorphism. ■

Theorem 4.1.2. Every distributive algebraic lattice is Pseudo complement.

Proof:

Let L be a distributive algebraic lattice. Then $L \cong I(S)$ for some distributive join semilattice S with 0, $I(L)$ is complete. Let $I, I_n \in I(S)$, we have to show that $I \wedge (\bigvee I_k) = \bigvee (I \wedge I_k)$ of course $\bigvee (I \wedge I_k) \subseteq I \wedge (\bigvee I_k)$ (i)

Let $x \in I \wedge (\bigvee I_k)$ then, $x \in I$ and $x \in \bigvee I_k$ implies that $x \leq I_{n_1} \vee \ldots\ldots\ldots\ldots\ldots\ldots\ldots I_{n_n}$ for some $I_{k_i} \in I_{k_1}, I_{k_2} \ldots\ldots\ldots\ldots\ldots I_{k_n} \in I_{k_n}$ implies that $x \in I_{k_1} \vee \ldots\ldots\ldots\ldots\ldots\ldots \bigvee I_{k_n}$
implies that $x \in I \wedge (I_{k_1} \vee \ldots\ldots\ldots\ldots\ldots\ldots \bigvee I_{k_n}) = (I \wedge I_{k_1}) \vee \ldots\ldots\ldots\ldots\ldots\ldots \bigvee (I \wedge I_{n_n})$
implies that $I \wedge (\bigvee I_k) \subseteq \bigvee (I \wedge I_k)$ (ii)

From (i) and (ii) $\bigvee (I \wedge I_k) = I \wedge (\bigvee I_k)$ implies that $I(S)$ holds JID implies that $I(S)$ is Pseudo complemented implies that L is Pseudo complemented . ■

Theorem 4.1.3. Let L be a Pseudo complemented meet semi-lattice. $S(L) = \{a^* / a \in L\}$. Then the partial ordering of L partially orders $S(L)$ and makes $S(L)$ into a Boolean lattice. For $a, b \in S(L)$ we have $a \wedge b \in S(L)$ and the join in $S(L)$ is described by $a \vee b = (a^* \wedge b^*)^*$.

Proof:

The following result have been proved in congruence part.

(i) $a \leq a^{**}$
(ii) $a \leq b$ implies that $a^* \geq b^*$
(iii) $a^* = a^{***}$
(iv) $a \in S(L)$ iff $a = a^{**}$
(v) $a,b \in S(L)$ implies that $a \wedge b \in S(L)$
(vi) For $a,b \in S(L)\ Sup_s\ (L) \{a,b\} = (a^* \wedge b^*)^*$

For $a, b \in S(L)$ define $a \vee b = (a^* \wedge b^*)^*$, then (v) and (vi) $< S(L); \wedge \vee >$ is a bounded lattice.

Since for $a \in S(L)$ $a \wedge a^* = 0$ and $(a \vee a^*) = (a^* \wedge a^{**})^* = 0^* = 1$ implies that $S(L)$ is complemented lattice. Now we need only to show that S(L) is distributive.
For $x, y, z \in S(L)$, $x \wedge z \leq x \vee (y \wedge z)$ and $y \wedge z \leq x \vee (y \wedge z)$ Therefore $x \wedge z \wedge (x \vee (y \wedge z))^* = 0$
implies that $x \wedge (z \wedge x \vee (y \wedge z))^* = 0$ implies that $z \wedge (x \vee (y \wedge z))^* \leq x^*$
Again $y \wedge z \wedge (x \vee (y \wedge z))^* = 0$
or $y \wedge (z \wedge x \vee (y \wedge z))^* = 0$
$\therefore z \wedge (x \vee (y \wedge z))^* \leq y^*$

We can write $z \wedge (x \vee (y \wedge z))^* \leq x^* \wedge y^*$ consequently, $z \wedge (x \vee (y \wedge z))^* \wedge (x^* \wedge y^*)^* = 0$, Which implies that $z \wedge (x^* \wedge y^*)^* \leq (x \vee (y \wedge z))^{**}$ Now the left hand side is $z \wedge (x \vee y)$ [by for $a, b \in S(L)$

$Sup_S(L)\{a,b\}=(a^* \wedge b^*)^*]$ and the right hand side is $x \vee (y \wedge z)$ [by $a \in S(L)$ iff $a=a^{**}$].
Thus we have $z \wedge (x \vee y) \leq x \vee (y \wedge z)$ which is distributive. ■

Definition: A distributive Pseudo complemented lattice L is called a stone lattice if $x^* \vee x^{**} = 1$ for each $x \in L$.

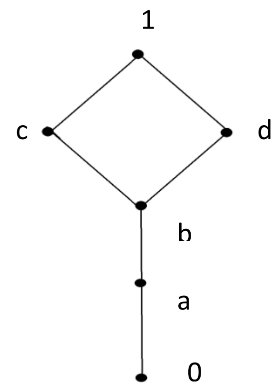

Fig. 4.4

A finite distributive lattice with only one atom is a stone lattice. A distributive lattice with Pseudo complementation L is called a stone algebra if and only if it satisfies the condition.

$$a^* \vee a^{**} = 1$$

which is called stone identity.

Theorem 4.1.4. A distributive Pseudo complemented lattice is a stone lattice iff $(a \vee b)^{**} = a^{**} \vee b^{**}$ for $a,b \in L$.

Proof:

Let L be a stone lattice then we have $(a \wedge b)^* = a^* \vee b^*$ for all $a, b \in L$.
Now $(a \vee b)^{**} = ((a \vee b)^*)^*$
$= (a^* \wedge b^*)^* = a^{**} \vee b^{**}$.

Conversely, let $(a \vee b)^{**} = a^{**} \vee b^{**}$ for $a, b \in L$.

Since L is a Pseudo complemented lattice, then for $a \in L$, $a \wedge a^* = 0$ implies that $(a \wedge a^*)^{**} = 0^{**}$ implies that $a^{**} \wedge a^{***} = 0$ implies that $a^{**} \wedge a^* = 0$

Now $(a \vee a^*)^* = a^* \wedge a^{**} = 0$ implies that $(a \vee a^*)^{**} = 0^*$

implies that $a^{**} \vee a^{***} = 1$

implies that $a^{**} \vee a^* = 1$

L is a stone lattice. ■

Theorem 4.1.5. Let L be a Pseudo complemented meet semi lattice and let $a,b \in L$ then $(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$.

Proof:

Since L is a Pseudo complemented meet semi lattice, then $a \leq a^{**}$ implies that $a \wedge b \leq a^{**} \wedge b$ implies that $(a \wedge b)^* \geq (a^{**} \wedge b)^*$ (i)

Again $b \leq b^{**}$ implies that $a^{**} \wedge b \leq a^{**} \wedge b^{**}$ implies that $a^{**} \wedge b \leq a^{**} \wedge b^{**}$

implies that $(a^{**} \wedge b)^* \geq (a \wedge b)^{***}$

implies that $(a^{**} \wedge b)^* \geq (a \wedge b)^*$ (ii)

From (i) and (ii) we have, $(a \wedge b)^* = (a^{**} \wedge b)^*$ (iii)

Again, $b \leq b^{**}$ implies that $a^{**} \wedge b \leq a^{**} \wedge b^{**}$

implies that $(a^{**} \wedge b)^* \geq (a^{**} \wedge b^{**})^*$ (iv)

Again, $a^{**} \leq a^{****}$ implies that $a^{**} \wedge b^{**} \leq a^{****} \wedge b^{**}$

$= (a^{**} \wedge b)^{**}$

implies that $(a^{**} \wedge b^{**})^* \geq (a^{**} \wedge b)^{***}$

implies that $(a^{**} \wedge b^{**})^* \geq (a^{**} \wedge b)^*$ (v)

From (iv) and (v)

$(a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$ (vi)

From (iii) and (vi)

$(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$. ■

Theorem 4.1.6. Let L be a Pseudo complemented distributive lattice, then for each $a \in L$, $(a]$ is Pseudo complemented distributive lattice in fact the Pseudo complement of $x \in (a]$ in $(a]$ is $x^* \wedge a$.

Proof:

Let $x \in (a]$ then $x \wedge (x^* \wedge a) = (x \wedge x^*) \wedge a = 0$ Further if $x \wedge t = 0$ then $t \leq x^*$ implies that $t \wedge a \leq x^* \wedge a$ implies that $t \leq x^* \wedge a$ implies that $x^* \wedge a$ is the Pseudo complement of x. Implies that $(a]$ is a Pseudo complemented distributive lattice. ■

Theorem 4.1.7. For a distributive lattice L with Pseudo complementation, the following conditions are equivalent.

(i) L is a stone algebra.

(ii) $(a \wedge b)^* = a^* \vee b^*$ for $a, b \in L$.

(iii) $a, b \in S(L)$ implies that $a \vee b \in S(L)$.

(iv) $S(L)$ is a subalgebra of L.

Proof:

(i) implies (ii)

Let L be a Stone algebra, we shall show that $a^* \vee b^*$ is the Pseudo complement of $a \wedge b$.
indeed, $(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*)$
$= (0 \wedge b) \vee (a \wedge 0)$
$= 0 \vee 0$
$= 0.$

If $(a \wedge b) \wedge x = 0$, then $(b \wedge x) \wedge a = 0$,
and so $b \wedge x \leq a^*$. Meeting both sides by a^{**} yields $b \wedge x \wedge a^{**} \leq a^* \wedge a^{**} = 0$;
that is, $b \wedge (x \wedge a^{**}) = 0$, implying that, $a^{**} \wedge x \leq b^*$

We have, $a^* \vee a^{**} = 1$, by Stone's identity.
$\therefore$ $x = x \wedge 1 = x \wedge (a^* \vee a^{**}) = (x \wedge a^*) \vee (x \wedge a^{**}) \leq a^* \vee b^*$.
implies that $a^* \vee b^*$ is the Pseudo complement of $a \wedge b$
implies that $(a \wedge b)^* = a^* \vee b^*$.

(ii) implies (iii)

Let $a, b \in S(L)$, then $a = a^{**}$, $b = b^{**}$
$\therefore a \vee b = a^{**} \vee b^{**} = (a^* \wedge b^*)^* = (a \vee b)^{**}$
implies that $a \vee b \in S(L)$.

(iii) implies (iv)

For $a, b \in S(L)$, $a \vee b \in S(L)$
Also $a^* = a^{**}$, $b^* = b^{**}$
Now, $a \vee b = a^{**} \vee b^{**} = (a^* \wedge b^*)^*$
$= (a \vee b)^{**}$
$= a \vee b$

i.e., $S(L)$ is a sub-algebra of L.

(iv) implies (i)

Let $S(L)$ is a sub algebra of L.
Then $a^* \vee a^{**} = (a \wedge a^*)^* = 0^* = 1$. Hence L is a Stone algebra. ■

4.2 Induction and congruence

Construction of Pseudo complemented lattice : An algebra, $< L;\wedge,\vee,*,0,1>$ where $\wedge$, $\vee$ are binary operation, $*$ is a unary operation, 0, 1 are nullary operations is called a lattice with Pseudo complementation if

(i) $< L;\wedge,\vee,0,1>$ is bounded lattice.

(ii) $*$ is a unary operation i.e. $\forall a \in L$ there exists a* such that $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \wedge a^* = x \;\; \forall x \in L$.

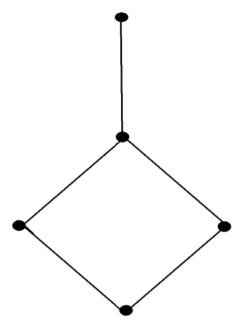

Fig 4.5

To see the difference in view point, consider the finite distributive lattice of Fig. (4.4). As a distributive lattice it has twenty five sublattice and eight congruences; as a lattice with Pseudo complementation it has three sub algebras and five congruences. *L* as lattice:

Sublattice: *{0},{a},{b},{c},{1},{0,a},{0,c},{0,b},{0,1},{0,a,b,c},L,{a,c},{a,c,1},{b,c},{b,c,1},{a,1},{b,1},{c,1},{0,a,1},{0,b,1},{0,c,1},{0,a,c},{0,b,c},{0,a,c,1},{a,b,c,1}* = 25

L as a lattice with Pseudo complementation *{0,1}, L, {0,c,1}*

Congruence : As a lattice,
ω = {0},{a},{b},{c},{1}
θ = {0,a},{b,c},{1}
φ = {0,a},{b,c,1}
Ψ = {0,b},{a,c},{1}
t = {0,b},{a,c,1}
ϵ = {0,a,b,c},{1}
ϑ={c,1},{a},{b},{0}
τ = {0,a,b,c,1}

Congruences as a lattice with Pseudo complementation.

$\omega, \varphi, t, \vartheta, \tau$

Theorem 4.2.1. Let L be a Pseudo complemented distributive lattice

$S(L) = \{a^*/a \in L\}$ *and* $D(L) = \{a/a^* = 0\}$ *then for* $a, b \in L$

(i) $a \wedge a^* = 0$

(ii) $a \leq b$ *implies that* $a^* \geq b^*$

(iii) $a \leq a^{**}$

(iv) $a^* = a^{***}$

(v) $(a \vee b)^* = a^* \wedge b^*$

(vi) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(vii) $a \wedge b = 0$ iff $a^{**} \wedge b^{**} = 0$

(viii) $a \wedge (a \wedge b)^* = a \wedge b^*$

(ix) $0^* = 1$ and $1^* = 0$

(x) $a \in S(L)$ iff $a = a^{**}$

(xi) $a,b \in S(L)$ implies that $a \wedge b \in S(L)$

(xii) $Sup_{S(L)} \{a,b\} = (a \vee b)^{**}$

(xiii) $0,1 \in S(L)$, $1 \in D(L)$ and $S(L) \cap D(L) = \{1\}$

(xiv) $a, b \in D(L)$ implies that $a \wedge b \in D(L)$

(xv) $a \in D(L)$ and $a \leq b$ imply that $b \in D(L)$

(xvi) $a \vee a^* \in D(L)$

(xvii) $x \rightarrow x^{**}$ is a meet – homomorphism of L onto $S(L)$

Proof :

(i) By the definition of Pseudo complement $a \wedge a^* = 0 \; \forall \; a \in L$

(ii) For $b \wedge b^* = 0$ and $a \leq b \Rightarrow a \wedge b^* = 0$ which implies $a^* \geq b^*$

(iii) By the definition of Pseudo complement $a \wedge a^* = a^* \wedge a = 0$. Similarly, $a^* \wedge (a^*)^* = 0 \Rightarrow a^* \wedge a^{**} = 0$ and $a^* \wedge a = 0 \Rightarrow a \leq a^{**}$, Hence $a \leq a^{**}$.

(iv) From (iii), $a \leq a^{**}$ implies that $a^* \geq a^{***}$ (A) [by (ii)]

Again $a^* \wedge a^{**} = 0$. i.e., $a^{**} \wedge a^* = 0$.

Similarly $a^{**} \wedge (a^{**})^* = 0$ implies that $a^{**} \wedge a^{***} = 0$

and $a^{**} \wedge a^* = 0$ implies that $a^* \leq a^{***}$ (B)

From (A) and (B)

we have $a^* = a^{***}$. Hence $a^* = a^{***}$.

(v) we have $(a \vee b) \wedge (a^* \vee b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*)$

$= (0 \wedge b^*) \vee (a^* \wedge 0)$ [by(i)]

$= 0 \vee 0$

$= 0$

Let $(a \vee b) \wedge x = 0$
implies that $(a \wedge x) \vee (b \wedge x) = 0$
implies that $a \wedge x = 0$ and $b \wedge x = 0$
implies that $x \leq a^*$ and $x \leq b^*$
implies that $x \leq a^* \wedge b^*$
Therefore $a^* \wedge b^*$ is the Pseudo complement of $a \vee b$.
Hence $(a \vee b)^* = a^* \wedge b^*$.

(vi) Let $a, b \in L$ implies that $a^*, b^* \in L$ implies that $a^{**}, b^{**} \in S(L)$ implies that $a^{**} \wedge b^{**} \in S(L)$.
But $a^{**} \wedge b^{**}$ is the smallest element of $S(L)$ containing $a \wedge b$. So $(a \wedge x)^{**} = a^{**} \wedge b^{**}$.

(vii) If $a \wedge b = 0$ by (vi) then $a^{**} \wedge b^{**} = (a \wedge b)^{**} = ()^{**} = ()$
So, $a^{**} \wedge b^{**} = 0$.
Conversely, if $a^{**} \wedge b^{**} = 0$ by (iii) $a \leq a^{**}, b \leq b^{**}$ $\forall\, a, b \in L$
then, $a \wedge b \leq a^{**} \wedge b^{**} = 0$
$\therefore a \wedge b = 0$. Hence $a \wedge b$ iff $a^{**} \wedge b^{**} = 0$.

(viii) Since $a \wedge b \leq b$ so $(a \wedge b)^* \geq b^*$ and so $a \wedge (a \wedge b)^* \geq a \wedge b^*$ (A)
Again $(a \wedge b) \wedge (a \wedge b)^* = 0$ implies that
$(a \wedge (a \wedge b)^*) \wedge b = 0$, therefore $a \wedge (a \wedge b)^* \leq b^*$
implies that $a \wedge a \wedge (a \wedge b)^* \leq a \wedge b^*$ implies that
$a \wedge (a \wedge b)^* \leq a \wedge b^*$ (B)
From (A) and (B) $a \wedge (a \wedge b)^* = a \wedge b^*$.
Hence $a \wedge (a \wedge b)^* = a \wedge b^*$.

(ix) We have $0 \wedge x = 0\ \forall x \in L$ and $0 \wedge 1 = 0$
But $x \leq 1\ \forall x \in L$. Hence $0^* = 1$
Again, $0^* = 1$ implies that $0^{**} = 1^*$ implies that $0 = 1^* \therefore 1^* = 0$.

(x) If $a \in S(L)$ then, $a = b^*$ for some $b \in L$, but $a^* = a^{***}, \forall a \in L$
Now, $a^{**} = b^{***} = b^* = a$ Hence, $a^{**} = a$
Conversely, if $a = a^{**}$ then $a = b^*$, thus $a \in S(L)$.
Hence $a \in S(L)$ iff $a = a^{**}$

(xi) Let $a, b \in S(L)$ then $a = a^{**}$, $b = b^{**}$.
Since $a \wedge b \leq a$ implies that $(a \wedge b)^{**} \leq a^{**} = a$ $\therefore a \geq (a \wedge b)^{**}$
Again, since $a \wedge b \leq b$ implies that $(a \wedge b)^{**} \leq b^{**} = b$
$\therefore (a \wedge b)^{**} \leq b$ implies that $b \geq (a \wedge b)^{**}$ implies that
$a \wedge b \geq (a \wedge b)^{**}$ (A)
But $a \wedge b \leq (a \wedge b)^{**}$ (B)
From (A) and (B) $a \wedge b = (a \wedge b)^{**}$ implies that $a \wedge b \in S(L)$.
If $x \in S(L)$ such that $x \leq a$ and $x \leq b$ then $x \leq a \wedge b$.
i.e., $a \wedge b$ is a greatest lower bound of $S(L)$.
Therefore $a \wedge b = Inf_{s(L)} \{a, b\} \in S(L)$.

(xii) For $a, b \in S(L)$ since $a^* \geq a^* \wedge b^*$ implies that
$a^{**} \leq (a^* \wedge b^*)^*$ [by (ii)] implies that $a \leq (a^* \wedge b)^*$ [by (i)]
Again $b^* \geq a^* \wedge b^*$ implies that $b^{**} \leq (a^* \wedge b^*)^*$ [by (ii)]
implies that $b \leq (a^* \wedge b^*)^*$ [by (i)]
$\therefore (a^* \wedge b^*)^*$ is a upper bound of $\{a, b\}$ in $S(L)$.
Let $x \in S(L)$ such that $a \leq x$, $b \leq x$ then $a^* \geq x^*$, $b^* \geq x^*$ [by (ii)]
$\therefore a^* \wedge b^* \geq x^*$ implies that $(a^* \wedge b^*)^* \leq x^{**} = x$
implies that $(a^* \wedge b^*)^* \leq x$.
$\therefore (a^* \wedge b^*)^*$ is a least upper bound of $\{a, b\}$ in $S(L)$
$Sup_{s(L)} \{a, b\} = (a^* \wedge b^*)^*$.
Again $(a \vee b)^{**} = ((a \vee b)^*)^* = (a^* \wedge b^*)^*$
Hence $Sup_{s(L)} \{a, b\} = (a \vee b)^{**} = (a^* \wedge b^*)^*$.

(xiii) From (ix) we have $0^* = 1$, $1^* = 0$ then $0,1 \in S(L)$ and $1 \in D(L)$. Let $x \in S(L) \cap D(L)$ then $x \in S(L)$ and $x \in D(L)$ such that $x = x^{**}$, $x^* = 0$
then $x = (x^*)^* = 0^* = 1$.
Hence $S(L) \cap D(L) = \{1\}$

(xiv) Let $a, b \in D(L)$ then $a^* = 0$, $b^* = 0$ implies that $a^{**} = b^{**} = 0^* = 1$.
Now, $(a \wedge b)^{**} = a^{**} \wedge b^{**} = 1 \wedge 1 = 1$. By (iv) $(a \wedge b)^* = (a \wedge b)^{***} = 1^* = 0$
implies that $a \wedge b \in D(L)$.

(xv) If $a \in D(L)$ then $a^* = 0$ and $a \leq b$ implies that $a^* \geq b^*$.
implies that $b^* \leq a^* = 0$ implies that $b^* = 0$. Hence $b \in D(L)$.

(xvi) From (v) we have $(a \vee a^*)^* = a^* \wedge a^{**} = a^* \wedge (a^*)^* = 0$.
Hence $a \vee a^* \in D(L)$.

(xvii) Let $\varphi: L \rightarrow S(L)$ defined by $\varphi(x) = x^{**}$.
Then, $\varphi(x \wedge y) = (x \wedge y)^{**} = x^{**} \wedge y^{**} = \varphi(x) \wedge \varphi(y)$.
$\therefore$ φ is a meet homomorphism . ■

Theorem 4.2.2. Any complete lattice that satisfies the Join Infinity Distributive Identity (JID) is a Pseudo complemented distributive lattice.

Proof :

Let L be a complete lattice. For $a \in L$ set $a^* = \bigvee (x|x\in L, a \wedge x = 0)$. Then, by (JID), $a \wedge x^* = a \wedge \bigvee (x|a \wedge x = 0) = \bigvee (a \wedge x|a \wedge x = 0) = \bigvee(0) = 0$ suppose $a \wedge x = 0$, then $x \leq a^*$ by the definition of a^*; thus a^* is the Pseudo complement of a and so L is Pseudo complemented.
■

Remark 4.2.3. The theorem shows that for stone algebra, the behavior of the skeleton and dense set is decisive. This conclusion leads us to form the goal of research for stone algebras and for all distributive lattice with Pseudo complementation.

Let L be a distributive lattice with Pseudo complementation. For a congruence relation Θ of L. Let Θ_S and Θ_D denote the restrictions of Θ to $S(L)$ and $D(L)$ respectively. Obviously, Θ_D is congruence relation of $D(L)$. In $S(L)$ the operations are $x \wedge y$, $x \vee y = (x^* \wedge y^*)^*$ and $*$, therefore Θ_S is clearly a congruence relation of $S(L)$. Thus $<\Theta_S,\Theta_D> \in C(S(L)) \times C(D(L))$.

An arbitrary pair $<\varphi ,\Psi> \in C(S(L)) \times C(D(L))$ will be called a congruence pair if $a \in S(L)$, $u \in D(L)$, $u \geq a$ and $a \equiv 1(\varphi)$ imply that $u = 1(\Psi)$.

Corollary 4.2.4. Let L be an arbitrary lattice, then $C(L)$ is an algebraic lattic.

Proof :

We already know that $C(L)$ is a complete distributive lattice. Suppose $\Theta \in C(L)$ observe that $\Theta =\bigvee(\theta(a, b)|a\equiv b\ \Theta, a, b \in L)$. Sinceevery principal congruence is compact, So $C(L)$ is algebraic.

Theorem 4.2.5. Let L be a distributive lattice with Pseudo complementation. Then every congruence relation Θ of L determines a congruence pair $<\Theta_S,\Theta_D>$.

Conversely every congruence pair $<\Theta_1,\Theta_2>$ uniquely determines a congruence relation Θ on L with $\Theta_S = \Theta_1$ and $\Theta_D = \Theta_2$ by the following rule $x \equiv y\ (\Theta)$ iff (i) $x^* \equiv y^*\ (\Theta_1)$ and (ii) $x \vee u \equiv y \vee u\ (\Theta_2)$ for all $u \in D(L)$.

Proof :

The first statement is obvious, let Θ be a congruence of L. $x, y \in L$, $x \equiv y\ (\Theta)$. By theorem, $x = x^{**} \wedge (x \vee x^*)$, $y = y^{**} \wedge (y \vee y^*)$ and $x^{**}=y^{**}(\Theta_S)$, $x \vee x^* = y \vee y^*\ (\Theta_D)$ thus Θ_S and Θ_D do indeed determine Θ.

Let $<\Theta_1,\Theta_2>$ be a congruence pair and let Θ be defined by (i) and (ii) Θ is obviously an equivalence relation. To show the substitution property for *, let $x \equiv y\ (\Theta)$. Then by (i), $x^* \equiv y^*\ (\Theta_1)$ and thus $x^{**} \equiv y^{**}(\Theta_1)$, which is (i) for x^* and y^*. Since $x^* \equiv y^*(\Theta_1)$ and $S(L)$ is Boolean. There is an $a \in S(L)$ such that $a=1(\Theta_1)$ and $x^* \wedge a \equiv y^* \wedge a\ (\Theta_1)$. Thus for any $u \in D(L)$. We obtained $u \vee a \equiv 1\ (\Theta_2)$ by the definition of the congruence pair, and so

$x^* \vee u \equiv (x^* \vee u) \wedge (a \vee u) = (x^* \wedge a) \vee u \equiv (y^* \wedge a) \vee u$
$= (y^* \vee u) \wedge (a \vee u) \equiv y^* \vee u\ (\Theta_2)$,

Proving (ii) for x^* and y^*. Therefore Θ is a congruence relation.

For $x, y \in S(L)$, $x \equiv y\ (\Theta)$ iff $x^* \equiv y^*\ (\Theta_1)$ (since (ii) is trivial), and so $x \equiv y(\Theta_S)$ iff $x \equiv y\ (\Theta_1)$, that is $\Theta_S = \Theta_1$. For $x, y \in D(L)$, (i) is trival and thus $x \equiv y\ (\Theta_1)$, iff for all $u \in D(L)$, we have $x \vee u \equiv y \vee u(\Theta_2)$, which is equivalent to $x \equiv y\ (\Theta_2)$, and so $\Theta_2 = \Theta_D$. ■

Lemma 4.2.6. Let L be an algebra and let Θ be a congruence relatin of L. For any congruence $\emptyset$ of L such that $\emptyset \geq \Theta$, define the relation $\emptyset/\Theta$ on L/Θ by $[x]\ \Theta \equiv [y]\ \Theta\ (\emptyset/\Theta)$ iff $x \equiv y(\emptyset)$.

Then $\emptyset/\Theta$ is a congruence of L/Θ. Conversely, every convergence $\Psi = \emptyset/\Theta$ for some congruence $\emptyset \geq \Theta$.

Proof :

We have to prove that $\emptyset/\Theta$ is well defined, (ii) is an equivalence relation, and (iii) has the substitution property. To represent Ψ define $\emptyset$ by $x \equiv y(\emptyset)$ iff $[x]\ \Theta \equiv [y](\Psi)$

Again, we have to verify that $\emptyset$ is a congruence $\emptyset/\Theta = \Psi$ follows from the definition of $\emptyset$.

Definition 4.2.7. A class K of algebras is said to have the congruence extension property if, for $A,B \in K$ with A a subalgebra of B and Θ a congruence of A, there exists a congruence $\emptyset$ on B such that $\emptyset_A = \Theta$, i.e $\emptyset$ restricted to A is Θ.

Remark 4.2.8. Using this terminology, the class of distributive lattices D has the Congruence Extension Property.

Theorem 4.2.9. (G. Grätzer and H. Lakser [a]). The class of all distributive lattices with Pseudo complementation enjoys the Congruence Extension Property.

Proof :

Let L and K be distributive lattices with Pseudo complementation, let L be a subalgebra of K, and let Θ be a congruence of L given by the congruence pair $<\Theta_1,\Theta_2>$. It is clear from 4.2.7 that we need only show the existence of a congruence pair $<\emptyset_1,\emptyset_2>$ of K such that $(\emptyset_1)_{S(L)} = \Theta_1$ and $(\emptyset_2)_{D(L)} = \Theta_2$.

Let $J_L = [1]\ (\Theta_1)$ and put $J_K = [\,J_L\,)$ the dual ideal generated by J_L in $S(K)$. Then $\emptyset_1$ can be defined as the congruence of $S(K)$ associated with J_K, that is, $[1]\ \emptyset_1 = J_K$. Set $I = \{i/i \in D(K)\}$, $i \geq u$ for some $u \in J_L$ }. Then I is a dual idesl of $D(K)$; in fact, $I = [\,J_K\,) \cap D(K)$. By the definition of congruence pair, we have to find a congruence $\emptyset_2$ on $D(K)$ such that $(\emptyset_2)_{D(L)} = \Theta_2$ and $[I]\emptyset_2 \geq I$, note that I has the following property:

If $u \in I$, $v \in D(L)$ and $v \leq u$, then there exists a $v_1 \in D(L)$, $v_1 \leq u$ such that $v_1 \in [1]\ \Theta_2$.

Indeed, $u \in I$ means that $u \geq x$ for some $x \in J_L$, and thus $v_1 = v \vee x$ will do the trick.

Summarizing, to complete the proof it suffices to prove the following statement:

Let A and B be distributive lattices with 1, A a {1}-sub-lattice of B, Θ a congruence of A, and I a dual of B satisfying the condition:

If $u \in I$, $v \in A$, and $v \leq u$, then $v_1 \leq u$ for some $v_1 \in [i]\Theta$. Then there exists a congruence relation $\emptyset$ on B satisfying $\emptyset_A = \Theta$ and $[\,I\,]\ \emptyset \supseteq I$.

To prove this statement, consider $\Theta\ [\,I\,]$ defined by the dual of a known corollary. If $a, b \in A$ and $a \equiv b\ (\Theta\ [\,I\,])$, then $a \wedge b = (a \vee b) \wedge i$ for some $i \in I$. Thus by our assumption on I, there is an $i_1 \in [i]\ \Theta$ such that $i_1 = I$. Therefore $a \wedge b = (a \vee b) \wedge i \geq (a \vee b) \wedge i_1 \equiv (a \vee b) \wedge I = a \vee b\ (\Theta)$, and so $a \equiv b(\Theta)$. Having shown that $(\Theta[I]_A \leq \Theta$, we can from $A/\Theta[I]_A$, $B/\Theta[I]$, and $\Theta[I]_A$ $/\Theta$. There exists a congruence Ψ on $B/\Theta[I]$ such that Ψ is restricted to $A/\Theta[I]_A$ is $\Theta[I]/\Theta$. By the previous lemma, there is a unique congruence $\emptyset$ or B such that $\emptyset/\Theta[I] = \Psi$ and $\emptyset \geq \Theta[I]$. Obviously, $\emptyset$ satisfies the requirements. ■

Theorem 4.2.10. For a Pseudo complemented distributive lattice L, define the relation R by: $x \equiv y(R)$ *iff* $x^* = y^*$. Then R is a congruence on L and $L \mid R \cong S(L)$.

Proof :

Given that $x \equiv y(R) <=> x^* = y^*$, then $x^* = x^*$ implies that $x \equiv x(R)$ implies that R is reflective. Also if $x \equiv y(R)$, then $x^* = y^*$ implies $y^* = x^*$ implies that $y = x(R)$ implies that R is symmetric. Let $x \equiv y(R)$ and $y \equiv z(R)$ then $x^* = y^*$ and $y^* = z^*$ implies that $y \equiv x(R)$ implies that R is symmetric. Let $x \equiv y(R)$ and $y \equiv z(R)$ then $x^* = y^*$ and $y^* = z^*$ implies that $x^* = z^*$ implies that $x \equiv z(R)$ implies that R is transitive. Implies that R is equivalence relation.

Now, suppose $x \equiv y(R)$ and $t \in L$ then $x^* = y^*$ implies that $x^{**} = y^{**}$.
Now $(x \wedge t)^{**} = x^{**} \wedge t^{**} = y^{**} \wedge t^{**} = (y \wedge t)^{**}$
implies that $(x \wedge t)^{**} = (y \wedge t)^{**}$
implies that $(x \wedge t)^* = (y \wedge t)^*$
implies that $x \wedge t = y \wedge t\ (R)$
and $(x \vee t)^* = x^* \wedge t^* = y^* \wedge t^* = (y \vee t)^*$
implies that $x \vee t = y \vee t\ (R)$.

So, R is congruence relation on L.

Define $\varphi: L \mid R \rightarrow S(L)$ by $\varphi((a]R) = a^{**}$
Then $\varphi([a] \wedge [b]) = \varphi([a \wedge b])$
$= (a \wedge b)^{**} = a^{**} \wedge b^{**}$
$= \varphi([a]) \wedge \varphi([b])$

and $\varphi([a] \vee [b]) = \varphi([a \vee b])$
$= (a \vee b)^{**} = (a^* \wedge b^*)^*$
$= (a^{***} \wedge b^{***})^* = a^{**} \vee b^{**}$
$= \varphi([a]) \vee \varphi([b])$
φ is a homomorphism .

To show that φ is one-one. Let $a^{**} = b^{**}$ implies that $a^* = b^*$ implies that $a \equiv b(R)$ implies that $[a] = [b]$.

$\therefore \varphi$ is one – one. let $a \in S(L)$ then $a = a^{**}$ implies that $a = \varphi([a])$ implies that φ is onto.
Hence $: L \mid R \rightarrow S(L)$ is an isomorphism. Therefore $L \mid R \equiv S(L)$. ■

Recommendations and Application

Conclusion and Future Recommendations: From the discussions of all previous chapters it can be concluded and recommended that the concept of Pseudo complemented with distributive lattice can be introduce in $I_n(L)$; $P_n(L)$ which are normal, relatively normal etc.

Application: Lattice theory has a lot of applications in different fields. Boolean lattice has applications in the field of hardware and software development of computer science. Also it has wide applications in networking. It can be applied to develop theories in other branches of algebra, such as group theory, Ring and Modules etc.

One of the major applications of Boolean lattices is in the switching systems, which are network of switches that involve two state devices 0 and 1 for off and on respectively.

References

1. Katrinak, *Congruence lattice of distributive p-algebras*, Algebra Universalis, 7(1977) 265-271.

2. G. Gratzer & E.T. Schmidt, *On a problem of M.H. Stone*, Acta.Math. Acad. Sci. Hung 8(1957).455-460

3. H Lasker, *Principal congruences of Pseudo complemented distributive lattices*. Proc, Amer. Math. Soc 37(1973). 32-36.

4. T.Katrinak & P.Medarly, *Constructions of p-algebras.* Algebra Universalis, 17(1983), 288- 316

5. T.Katrinak & P.Medarly, *The structure of Pseudo complemented distributive lattices*, Proc. Amer.Math.Soc. 37(1973), 32-36

6. K.B Lee, *Equations classes of a distributive Pseudo complemented lattices,* Canad J. math. 22(1970), 881-891.

7. A.S.A Noor& M.A Latif, *Permutability of standard n-congruences*, Rajshahi University Studies (Part B) Journal of science Vol. 23-24 (1995-96) 211-214

8. W.H Cornish, *The multiplier entension of distributive lattice*. J Algebra 32(1974) 339-355.

9. A Davey, *Some annihilator conditions on distributive lattice*s Alg. Univ 3(4) (1974) 316-322.

10. G. Gratzer, *Lattice theory, First Concepts an distributive lattices*. Freeman San Francisco 1971.

11. Vijay K Khanna *Lattices and Boolean algebras* [First Concepts]

12. R. Balbes & A Horn, *Stone Lattices*, Duke Math. J 38(1971) 537-546.

13. C.C Chen & G. Gratzer, *Stone lattices construction theorems*, cand, J.Math.21 (1969) 884-894

14. C.C Chen & G. Gratzer, *Stone lattices, π: Structure*, theorems, cand J.Math 21(1969) 895-903

15. W.H Cornish, *Normal lattices,* J Austral Math Soc.14(1972) 200-215.

16. W.H Cornish, *On a problem of M.H Stone Acta Math*. Acad.Sci Hung 8(1957) 455-460

17. J.E Kist, *Minimal prime ideals in communicative semigroups*, Proc. London Math Soci (3) (1983)

18. T.P Speed, *On Stone Lattices*. J Austral. Math. Soc 9(1969) 297-307

19. J. Varlet, *On the characterization of stone lattices*, Acta Sci. Math(szeged) 27(1966) 81-84.

20. T. Katrinak, *Remark on stone lattices I (Russian) Math FYZ Cusopis 16(1966) 128-142*

21. T. Katrinak , *A new proof of the construction theorem for stone algebra*, proc. Amer Math.Soc 40(1973) 75-78

22. *A new description of the free stone algebras*, Algebra univ. 5(1975) 179-189.

23. G. Gratzer & E.T. Schmidt, *Notes of stone lattices II(Russian) Math*. Casopis. Sloven. Akad Vied 17(1967) 20-37