Analytical Solution of Quadratic Nonlinear Oscillator by Extended Iteration Method

by

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Dedicated to My Family

Father: MD. Mobarok Hossain

Mother: Momotaj Irin

Sisters: Sabrina Akter Mitu,

Sumiya Noushin Oyshe

&

Beloved Husband: S. M. Masud Rana

Declaration

This is to certify that the thesis work entitled “Analytical Solution of Quadratic Nonlinear Oscillator by Extended Iteration Method” has been carried out by Shajia Afrin Flora in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Signature of Supervisor   Signature of Student

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Abstract

A modified approximate analytic solution of the quadratic nonlinear oscillator "$\ddot{x}+x^2=0$" has been obtained based on an Extended Iteration method. In this study the Fourier series and utilized indispensable truncated terms have been used in each step of Extended Iterations. The approximate frequencies obtained by this technique show a good agreement with the exact frequency. The percentage of error between exact frequency and our third approximate frequency is as low as 0.001%.There is no algebraic complexity in our calculation that is why this technique is very easy. The results have been compared with the exact results and other existing results that are convergent as well as consistent.

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CHAPTER I

Introduction

Nonlinear dynamic problems have fascinated the applied mathematicians, physicists and engineers from a long time. Over the past few decades applications in solid and structural mechanics as well as fluid mechanics have appeared, and Now-a-days there is a widespread interest in nonlinear oscillators, strange attractors, chaotic and dynamical systems theories in the engineering and applied science communities.

Physical and mechanical oscillatory systems are often governed by the nonlinear differential equations. Unfortunately, with the exception of a number of particular cases, the exact analytical solutions of such equations cannot be determined. In many cases, it is possible to replace the nonlinear differential equation by a corresponding linear differential equation that approximates the original nonlinear equation closely to give useful results. Often such linearization is not feasible or possible and for this situation the original differential equation itself must be directly dealt with.

However, in many cases it is possible to compute accurate approximate analytical solutions of the equations. There are a large number of approximate methods commonly used for solving nonlinear oscillatory systems such as Perturbation, Harmonic Balance (HB), Homotopy Perturbation, Homotopy Analysis, Iteration, Extended Iteration etc. The Perturbation method is mainly used for the small nonlinear problems. On the other hand, Harmonic Balance and Iteration methods are mostly used for the strong and as well as small nonlinear problems.

One important class of nonlinear oscillators are conservative oscillators in which the restoring force is not dependent on time, the total energy is constant and any oscillation is stationary. In spite of the great elegance and simplicity of such equations, the solutions of specific problems are significantly hard to derive. Finding innovative method to analyze and solve these equations has become an interesting subject in the field of ordinary and partial differential equations and dynamical systems. The nonlinear equations in most of the real-life problems are not always possible and sometimes not even advantageous to express exact solutions of nonlinear differential equations explicitly in terms of elementary functions or independent spatial and/or temporal variables; however, it is possible to find approximate solutions.

Perturbation means grossly small change so the method is adopted when the nonlinearity is small. Thus in case of strong nonlinearities, Perturbation method is not generally adopted. It is used to construct uniformly valid periodic solution to second-order nonlinear differential equations. A critical feature of the technique is a middle step that breaks the problem into "solvable" and "Perturbation" parts. Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

Harmonic Balance method is a procedure of determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. An important advantage of the method that can be applied to nonlinear oscillatory problems for which the nonlinear terms are not “small” i.e., no Perturbation parameter need exist. A disadvantage of the method that is difficult prior to predict for a given nonlinear differential equation whether a first order Harmonic Balance calculation will provide a sufficiently accurate approximation to periodic solution.

Iterative technique is particular a technique for calculating approximate periodic solutions and corresponding frequencies of truly nonlinear oscillators for small and as well as large amplitude of oscillation.

The main intention of this thesis is to investigate the approximate analytic solutions using the modified Extended Iterative method to decompose the secular term, so that the solution can be obtained by Iterative procedure. This means that we can use Extended Iterative method to investigate many nonlinear problems. The main thrust of this technique is that the obtained solution rapidly converges to the exact solutions.

The chapter outline of this thesis is as follows**:** In **Chapter II**, the review of literature is presented. In **Chapter III,** the modified Extended Iterative method has been described for obtaining approximate analytical solutions of quadratic nonlinear oscillator. In **Chapter IV,** the results of the adopted method have been shown. Finally, In **Chapter V,** some conclusions and recommendations are included.

CHAPTER II

Literature Review

The review of literature is presented in this chapter. Here some general techniques described that can be used to illustrate the existence of periodic solutions for a given truly nonlinear equation. These methods also apply to the case of standard equation. Moreover, this chapter shows some existing methods and their problem solving procedure, which help us in comparative analysis.

2.1 Introduction

The study of nonlinear problems is one of most striking parts in mathematics, physics and other science and engineering. So mathematicians, physicists, engineers and others scientist are of interest to nonlinear problems. A system of nonlinear equations is a set of simultaneous equations in which the unknowns appear as variables of a polynomial of degree higher than one or in the argument of a function which is not a polynomial of degree one. On the other side, in a system of nonlinear equations, the equations to be solved cannot be written as a linear combination of the unknown variables or functions that appear in it or them. It does not matter when the nonlinear known functions appear in the equations. Particularly, a differential equation is regarded as linear if it gets linear in terms of the unknown function as well as its derivatives, even if nonlinear in terms of the other variables appearing in it.

2.2 Description of the Different Methods

Nonlinear equations are difficult to explore and nonlinear systems are commonly approximated by linear equations. This works well up to some accuracy and some range for the input values, but some interesting phenomena such as chaos and singularities are hidden by linearization. It follows that some aspects of the behavior of a nonlinear system appear commonly to be chaotic, unpredictable or counterintuitive. Although such chaotic behavior may resemble random behavior, it is absolutely not random. In this position there are different analytical approaches to find approximate solutions to nonlinear problems, such as: Perturbation method (Nayfeh A H, 1973, 1981; Rahman *et al.*, 2009; Alam *et al.*,2011; Haque *et al.*, 2011; Rahman *et al.*, 2011), Homotophy Perturbation method (Belendez *et al.*, 2007(a); Beléndez *et al.*, 2008(b); Belendez *et al.*, 2009(a),(b)), Harmonic Balance method (Mickens R E, 1961, 1984, 1998, 2001, 2007; Hu H, 2006; Hu and Tang , 2006(a); Alam *et al.*, 2007; Beléndez and Pascual, 2007; Belendez *et al.*, 2007(b); Belendez *et al.*, 2008(a); Belendez *et al.*, 2009(c)), Modified Linstedt-Poincare method (He J H, 2001), Krylov-Bogoliubov-Mitropolskii (KBM) method (Krylov and Bogoliubov, 1947; Bogoliubov *et al.*, 1961), Energy Balance method (Ozis and Yildrim, 2007), Cubication method (Belendez *et al.*, 2009(d)), Iterative method (Mickens R E, 1987, 2005, 2010; Lim and Wu, 2002; Hu H, 2006(a),(b); Hu and Tang, 2006(b); Chen and Liu, 2008; Haque *et al.*, 2013; Haque B M I, 2013; Haque B M I, 2014; Haque *et al.*, 2014; Haque and Hossain, 2016; Haque *et al.*, 2016(a); Haque *et al.*, 2016(b); Haque *et al.*, 2016(c); Haque *et al.*, 2017) etc.

Almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This is so called small parameter assumption greatly restrict application of perturbation techniques. The perturbation method is the most widely utilized method in which the nonlinear term is small. The method of Lindstedt-Poincare (He J H, 2001), Krylov-Bogoliubov-Mtropolskii (KBM) method (Krylov and Bogoliubov, 1947; Bogoliubov *et al.*, 1961),Multiple Scales method (Lakrad and Belhaq, 2002), and Homotopy Perturbation method (Belendez *et al.*, 2007(a); Beléndez *et al.*, 2008(b); Belendez *et al.*, 2009; Belendez *et al.*, 2009(a),(b)) are most momentous among all Perturbation methods.

The method of Lindstedt-Poincare (He J H, 2001) is an introductory method to solved the following second order nonlinear differential equations

$$\ddot{x} + \omega_0{}^2 x + \varepsilon\, f(\ddot{x}, x) = 0\,, \tag{2.1}$$

where ω_0 is the unperturbed frequency and ε is a small parameter.

The fundamental idea in Lindstedt's technique is based on the observation that the nonlinearities alter the frequency of the system from the linear one ω_0 to $\omega(\varepsilon)$. To account for this change in frequency, he introduces a new variable $\tau = \omega t$ and expand ω and x in power of ε as

$$\begin{cases} x = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots \\ \omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots, \end{cases} \tag{2.2}$$

where $\omega_i,\ i = 0,1,2,\dots$, are unknown constants to be determined.

Substituting equation (2.2) into equation (2.1) and equating the coefficients of the various powers of ε, the following equations are obtained

$$\begin{cases} \ddot{x}_0 + x_0 = 0 \\ \ddot{x}_1 + x_1 = -2\omega_1 \ddot{x} - f(x_0, \dot{x}_0) \\ \ddot{x}_2 + x_2 = -2\omega_1 \ddot{x}_1 - f(x_0,\ \dot{x}_0) - (\omega_1^2 + 2\omega_2)\ddot{x}_0 \\ \qquad\qquad - f_x(x_0,\ \dot{x}_0)x_1 + f_{\dot{x}}(x_0,\ \dot{x}_0)(\omega_1 \dot{x}_0 + \dot{x}_1) \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \ddot{x}_n + x_n = g_n(x_0, x_1, \dots, x_{n-1}; \dot{x}_0, \dot{x}_1, \dots, \dot{x}_{n-1}), \end{cases} \tag{2.3}$$

where over dot represents the differentiation with respect to τ.

Pointedly equation (2.3) is a linear system and it is solved by the elementary techniques.

This method is used only for finding the periodic solution, but the method cannot discuss transient case.

Therewithal, Krylov and Bogoliubov (1947) introduced a technique to discuss transients of the same equation. This method starts with the solution of the linear equation, assuming that, in the nonlinear case, the amplitude and phase in the solution of the linear equation are time dependent function rather than constants (Nayfeh A H, 1973). The solution of corresponding unperturbed equation (i.e., for $\varepsilon = 0$) of equation (2.1) can be written as

$$x = a\cos\left(\omega_0 t + \theta\right), \tag{2.4}$$

where a and θ are two arbitrary constants to be determined from the initial conditions $x(0) = x_0$ and $\dot{x}(0) = y_0$ which are respectively called amplitude and phase.

Now to determine an approximate solution of equation (2.1) for ε small but different from zero, Krylov and Bogoliubov (1947) assumed that the solution is still given by equation (2.4) with varying a and θ subjected to the conditions

$$\frac{dx}{dt}=-a\omega_0\sin\phi,\ \phi=\omega_0 t+\theta \tag{2.5}$$

Differentiating equation (2.4) with respect to time, t and using equation (2.5), we obtain

$$\left\{\frac{da}{dt}\cos\phi-\frac{d\theta}{dt}a\sin\phi=0.\right. \tag{2.6}$$

Again differentiating equation (2.5) with respect to time, t, we obtain

$$\frac{d^2x}{dt^2}=-a\,\omega_0^2\cos\phi-\omega_0\frac{da}{dt}\sin\phi-a\omega_0\frac{d\theta}{dt}\cos\phi. \tag{2.7}$$

Substituting equation (2.7) into equation (2.1) and using equation (2.4) and equation (2.5), we obtain

$$\frac{da}{dt}\omega_0\sin\phi+\frac{d\theta}{dt}a\omega_0\cos\phi=-\varepsilon\, f\left(a\cos\phi,\,-a\omega_0\sin\phi\right). \tag{2.8}$$

Solving equation (2.6) and equation (2.8), $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ yields

$$\begin{cases}\dfrac{da}{dt}=-\dfrac{\varepsilon}{\omega_0}\sin\phi\, f\left(a\cos\phi,\,-a\omega_0\sin\phi\right)\\[2ex] \dfrac{d\theta}{dt}=-\dfrac{\varepsilon}{a\omega_0}\cos\phi\, f\left(a\cos\phi,\,-a\omega_0\sin\phi\right).\end{cases} \tag{2.9}$$

Here equation (2.4) together with equation (2.9) represents the first approximate solution of equation (2.1).

Further, the technique was modified and justified by Bogoliubov and Mitropolskii (Bogoliubov *et al.*, 1961). They assumed a solution of the nonlinear differential equation (2.1) of the form

$$x(t,\varepsilon)=a\cos\psi+\varepsilon\, x_1(a,\psi)+\cdots+\varepsilon^n\, x_n(a,\psi)+O\left(\varepsilon^{n+1}\right), \tag{2.10}$$

where x_k, $k=1,2,\cdots,n$ is a periodic function of ψ with period 2π, a and ψ very with time, t according to

$$\begin{cases} \dfrac{da}{dt} = \varepsilon A_1(a) + \cdots + \varepsilon^n A_n(a) + O\left(\varepsilon^{n+1}\right) \\ \dfrac{d\psi}{dt} = \omega_0 + \varepsilon B_1(a) + \cdots + \varepsilon^n B_n(a) + O\left(\varepsilon^{n+1}\right), \end{cases} \tag{2.11}$$

where the function x_k, A_k and B_k are chosen such that equation (2.10) and equation (2.11) satisfy the differential equation (2.1). Later this solution was used by Mitropolskii Y (1964) to investigate similar system (i.e., equation (2.1)) in which the coefficient very slowly with time. Popov (Popov I P, 1956) extended this method to nonlinear strongly damped oscillatory systems. By Popov's (Popov I P, 1956) technique, Murty *et al.*, (1969) extended the method to over damped nonlinear system. Murty I S N (1971) further presented a unified KBM method to obtain under and over-damped solution of a second-order nonlinear differential equation. Shamsul and Sattar (Shamsul and Sattar, 1997) extended Murty I S N (1997) unified KBM method to investigate a third-order nonlinear differential equation.

Harmonic Balance method is the most useful technique for finding the periodic solutions of nonlinear system. Which is patented by Mickens (1961) and farther work has been done by Hu H (2006), Beléndez *et al.*, (2009)(c); Lim *et al.*, (2005), Wu *et al.*(2006) and so on for investigating the strong nonlinear problems. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series, whose coefficients are determined by requiring the series to satisfy the equation of motion. However, in order to avoid investigating an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of trigonometric function. This is the main task of Harmonic Balance method. Thus approximate solutions of an oscillator are obtained by Harmonic Balance method using a suitable truncated Fourier series.

The method is capable to determining analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter need exist. The formulation of the method of Harmonic Balance focuses primarily by Mickens (1984). However, it should be indicated that various generalizations of the method of Harmonic Balance has been made by an intrinsic method of harmonic analysis. Lately, combining the method of averaging and Harmonic Balance, Lim & Lai (2006) presented analytic technique to obtain first approximate Perturbation solution; their solution gives desired results for some non-conservative systems when the damping force is very small..

Mickens (2010) has given the general procedure for calculating solutions by means of the method of direct Harmonic Balance as follows:

He considered the equation for all Truly Nonlinear (TNL) oscillators as

$$F(x,\dot{x},\ddot{x})=0, \tag{2.12}$$

where $F(x,\dot{x},\ddot{x})$ is of odd-parity, i.e.,

$$F(-x,-\dot{x},-\ddot{x})=-F(x,\dot{x},\ddot{x}). \tag{2.13}$$

A major consequence of this property is that the corresponding Fourier expansions of the periodic solutions only contain odd harmonics, i. e.,

$$x(t)=\sum_{k=1}^{\infty}\left\{A_k\cos\left[(2k-1)\Omega t\right]+B_k\sin\left[(2k-1)\Omega t\right]\right\}. \tag{2.14}$$

The $N-th$ order Harmonic Balance approximation to $x(t)$ is the expression

$$x_N(t)=\sum_{k=1}^{N}\left\{\bar{A}_k^N\cos\left[(2k-1)\bar{\Omega}_N t\right]+\bar{B}_k^N\sin\left[(2k-1)\bar{\Omega}_N t\right]\right\}, \tag{2.15}$$

where $\bar{A}_k^N$, $\bar{B}_k^N$, $\bar{\Omega}_N$ are approximations to A_k, B_k, Ω for $k=1, 2, 3, ..., N.$

For the case of a conservative oscillator, equation (2.12) generally takes the form

$$\ddot{x}+f(x,\lambda)=0, \tag{2.16}$$

where λ denotes the various parameters appearing in $f(x,\lambda)$ and $f(-x,\lambda)=-f(x,\lambda)$.

The following initial conditions are selected

$$x(0)=A, \quad \dot{x}(0)=0, \tag{2.17}$$

and this has the consequence that only the cosine terms are needed in the Fourier expansions, and therefore we have

$$x_N(t)=\sum_{k=1}^{N}\bar{A}_k^N\cos\left[(2k-1)\bar{\Omega}_N t\right]. \tag{2.18}$$

Observe that $x_N(t)$ has $(N+1)$ unknowns, the N coefficients, and Ω_N, the angular frequency. These quantities may be calculated by carrying out the following ways:

Substitute equation (2.18) into equation (2.16), and expand the resulting form into an expression that has the following structure

$$\sum_{k=1}^{N} H_k \cos\left[\left(2k-1\right)\Omega_N\, t\right] + HOH \cong 0, \text{ HOH= Higher Order Harmonic} \tag{2.19}$$

where they H_k are functions of the coefficients, the angular frequency, and the parameters, i.e.,

$$H_k = H_k(\overline{A}_1^N,\ \overline{A}_2^N, \ldots\ldots, \overline{A}_N^N,\ \Omega_N,\ \lambda). \tag{2.20}$$

Here in equation (2.19), we only retain as many harmonics in our expansion as initially occur in the assumed approximation to the periodic solution. Set the functions H_k to zero, i.e.,

$$H_k = 0, \quad k = 1,\ 2, \ldots\ldots, N. \tag{2.21}$$

The action is justified since the cosine functions are linearly independent, as a result any linear sum of them that is equal to zero must have the property that the coefficient are all zero.

Solve the N equations in equation. (2.21), for $(\overline{A}_2^N, \overline{A}_3^N, \ldots\ldots \overline{A}_N^N)$ and Ω_N, in terms of $\overline{A}_1^N$. using the initial conditions, equation (2.17), we have for $\overline{A}_1^N$ the relation

$$x_N(0) = A = \overline{A}_1^N + \sum_{k=2}^{N} \overline{A}_k^N(\overline{A}_1^N,\ \lambda). \tag{2.22}$$

An important point is that equation (2.21) will have many distinct solutions and the one selected for a particular oscillator equation is that one for which we have known a priori restrictions on the behavior of the approximations to the coefficients. However, as the worked examples in the next section demonstrate, in general, no essential difficulties arise.

For the case of non-conservative oscillators, where $\dot{x}$ appears to an odd power the calculation of approximations to periodic solutions follows a procedure modified for the case of conservative oscillators presented above. Many of these equations take the form

$$\ddot{x} + f(x,\ \lambda_1) = g(x,\ \dot{x},\ \lambda_2)\dot{x}, \tag{2.23}$$

where

$$\begin{cases} f(-x,\ \lambda_1) = -f(x,\ \lambda_1) \\ g(-x,\ -\dot{x},\ \lambda_2) = -g(x,\ \dot{x},\ \lambda_2), \end{cases} \tag{2.24}$$

and $(\lambda_1,\ \lambda_2)$ denote the parameters appearing in $f(x,\ \lambda_1)$ and $g(x,\ \dot{x},\ \lambda_2)$.

For this type of differential equation, a limit-cycle may exist and the initial conditions cannot, in general, be a priori specified.

Harmonic balancing, for systems where limit-cycles may exist, uses the following procedures:

The $N-th$ order approximation to the periodic solution to be

$$x_N(t) = \bar{A}_1^N \cos\left(\bar{\Omega}_N t\right) + \sum_{k=2}^{N}\left\{ \bar{A}_k^N \cos\left[\left(2k-1\right)\bar{\Omega}_N t\right] + \bar{B}_k^N \sin\left[\left(2k-1\right)\bar{\Omega}_N t\right]\right\}, \quad (2.25)$$

where the $2N$ unknowns $\bar{A}_1^N, \bar{A}_2^N, ..., \bar{A}_N^N$; $\bar{\Omega}_N, \bar{B}_2^N, ..., \bar{B}_N^N$ and $\bar{\Omega}_N$ are to be determined.

Substitute Eq. (2.25) into Eq. (2.23) and write the result as

$$\sum_{k=1}^{N}\left\{ H_k \cos\left[\left(2k-1\right)\Omega_N t\right] + L_k \sin\left[\left(2k-1\right)\Omega_N t\right]\right\} + HOH \cong 0, \quad (2.26)$$

where the $\{H_k\}$ and $\{L_k\}$, $k=1$ to N, are functions of the $2N$ unknowns which are mentioned above.

Next equate the $2N$ functions $\{H_k\}$ and $\{L_k\}$ to zero and solve them for the $(2N-1)$ amplitudes and the angular frequency. If a valid solution exists, then it corresponds to a limit-cycle. In general, the amplitudes and angular frequency will be expressed in terms of the parameters λ_1 and λ_2.

Mickens (2010) has presented the following example**:**

Let us consider the nonlinear oscillator given by

$$\ddot{x} + x^3 = 0, \ x(0) = A, \ \dot{x}(0) = 0 \quad (2.27)$$

This approximation takes the form

$$x_1(t) = A\cos(\Omega_1 t) \quad (2.28)$$

Observe that this expression automatically satisfies the initial conditions. Substituting equation (2.28) into equation (2.27) gives $(\theta = \Omega_1 t)$

$$\left(-A\Omega_1^{\ 2}\cos\theta\right) + \left(A\cos\theta\right)^3 \cong 0,$$

$$-\left(A\Omega_1^{\ 2}\right)\cos\theta + A^3\left[\left(\frac{3}{4}\right)\cos\theta + \left(\frac{1}{4}\right)\cos 3\theta\right] \cong 0$$

$$A\left[-\Omega_1{}^2+\left(\frac{3}{4}\right)A^2\right]\cos\theta+HOH\cong 0$$

Setting the coefficient of $\cos\theta$ to zero gives the first approximation to the angular frequency

$$\Omega_1(A)=\left(\frac{3}{4}\right)^{1/2}A \tag{2.29}$$

and $$x_1(t)=A\cos\left[\left(\frac{3}{4}\right)^{1/2}At\right] \tag{2.30}$$

The solution for the second approximation takes the form $(\theta=\Omega_2 t)$

$$x_2(t)=A_1\cos\theta+A_2\cos 3\theta \tag{2.31}$$

with $$\ddot{x}_2(t)=-\Omega_2^2(A_1\cos\theta+9A_2\cos 3\theta) \tag{2.32}$$

Substituting equation (2.31) and equation (2.32) into equation (2.27), we obtain

$$H_1(A_1,A_2,\Omega_2)\cos\theta+H_2(A_1,A_2,\Omega_2)\cos 3\theta+HOH\cong 0,$$

where $$H_1=A_1\left[\Omega_2^2-\left(\frac{3}{4}\right)A_1^2-\left(\frac{3}{4}\right)A_1A_2-\left(\frac{3}{2}\right)A_2^2\right] \tag{2.33}$$

and $$H_2=-9A_2\Omega_2^2+\left(\frac{1}{4}\right)A_1^3+\left(\frac{3}{2}\right)A_1^2A_2+\left(\frac{3}{4}\right)A_2^3 \tag{2.34}$$

Setting H_1 to zero, and defining z as

$$z\equiv\frac{A_2}{A_1} \tag{2.35}$$

We obtain,

$$\Omega_2=\left(\frac{3}{4}\right)^{1/2}A_1\left(1+z+2z^2\right)^{1/2}=\Omega_1\left(1+z+2z^2\right)^{1/2} \tag{2.36}$$

where Ω_1 is that of equation (2.29). Inspection of equation (2.36) shows that the second approximation for the angular frequency is a modification of the first approximation result.

If this value for Ω_2 is substituted into equation (2.34) and this expression is set to zero, and if the definition of z is used, then the following cubic equation must be satisfied by z

$$51z^3 + 27z^2 + 21z - 1 = 0$$

There are three roots, but the one of interest should be real and have a small magnitude, i. e.,

$$|z| << 1$$

The root is $z_1 = 0.044818....,$

And implies that the amplitude, A_2, of the higher harmonic, i.e., the $\cos 3\theta$, is less than 5% of the amplitude of the fundamental mode, $\cos\theta$.

Therefore, the second harmonic balance approximation for equation (2.27) is

$$x_2(t) = A_1[\cos\theta + z_1 \cos 3\theta].$$

For the initial condition, $x_2(0) = A$, we obtain

$$A = A_1(1 + z_1) \text{ or } A_1 = \frac{A}{1+z_1} = (0.9571)A$$

Using the value of A_1 and z_1 into equation (3.35), we obtain

$$\Omega_2(A) = \left(\frac{3}{4}\right)^{1/2} A\left[\frac{\left(1 + z_1 + 2z_1^{\,2}\right)^{1/2}}{1+z_1}\right] = (0.8489)A$$

Hence, the second order harmonic balance approximation for the periodic solution of equation (2.27) is

$$x_2(t) = \left(\frac{A}{1+z_1}\right)[\cos(\Omega_2 t) + z_1 \cos(3\Omega_2 t)]$$

where z_1 and Ω_2 are given above equation.

Recently some authors used iterative technique (Mickens R E, 1987, 2005, 2010; Lim and Wu, 2002; Hu H, 2006(a),(b); Hu and Tang, 2006(b); Chen and Liu, 2008; Haque *et al*., 2013; Haque BM I, 2013; Haque BM I, 2014; Haque *et al*., 2014; Haque and Hossain, 2016; Haque *et al*., 2016(a); Haque *et al*., 2016(b); Haque *et al*., 2016(c); Haque *et al*., 2017) for calculating approximations to the periodic solutions and corresponding frequencies of TNL oscillator differential equations for small and as well as large amplitude of oscillation. The method was originated by Mickens in 1987. In the paper, he provided a general basis for iteration methods as they are currently used in the calculation of approximations to the periodic solutions of various nonlinear oscillatory differential equation successfully.

Mickens (2010) has given the general procedure for calculating solutions by means of the method of direct Iterative method as follows:

Step-1. Assume that the differential equation of interest is

$$F(\ddot{x}, x) = 0, \;\; x(0) = A, \; \dot{x}(0) = 0, \tag{2.37}$$

and further assume that it can be rewritten to the form

$$\ddot{x} + f(\ddot{x}, x) = 0, \tag{2.38}$$

Step-2. Next, add $\Omega^2 x$ to both sides to obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x, \ddot{x}) \equiv G(x, \ddot{x}), \tag{2.39}$$

where the constant Ω^2 is currently unknown.

Step-3. Now, formulate the Iterative scheme in the following way

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \quad k = 0, 1, 2, \ldots \tag{2.40}$$

with

$$x_0(t) = A\cos(\Omega_0 t), \tag{2.41}$$

such that the x_{k+1} satisfy the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{2.42}$$

Step-4. At each stage of the Iterative, k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

Step-5. This procedure gives a sequence of solutions: $x_0(t),\ x_1(t),\ \ldots$. Since all solutions are obtained from investigating linear equations, they are , in principle, easy to calculate. The only difficulty might be the algebraic intensity required to complete the calculations.

At this point, the following observations should be noted:

i. The solution for $x_{k+1}(t)$ depends on having the solutions for k less than $(k+1)$

ii. The linear differential equation for $x_{k+1}(t)$ allows the determination of Ω_k by the requirement that secular terms be absent. Therefore, the angular frequency, Ω appearing on the right-hand side of equation (2.38) in the function $x_k(t)$, is k.

iii. In general, if equation (2.38) is of odd parity, i.e.,

$$f(-\ddot{x},-x) = -f(\ddot{x},x), \tag{2.43}$$

then the $x_k(t)$ will only contain odd multiples of the angular frequency.

Here we present an Example performed by Mickens with the Direct Iterative (Lakrad and Belhaq, 2002), method:

Let us consider the oscillator

$$\ddot{x} + x^3 = 0,\ x(0) = A, \dot{x}(0) = 0 \tag{2.44}$$

and initial condition

$$x_0(t) = A\cos(\Omega_0 t) \tag{2.45}$$

A possible iteration scheme for this equation is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^3. \tag{2.46}$$

For k=0, we have

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 x_0 - x_0^3 = \Omega_0^2 (A\cos\theta) - (A\cos\theta)^3$$
$$= \left[\Omega_0^2 - \left(\frac{3}{4}\right) A^2 \right] A\cos\theta - \left(\frac{A^3}{4}\right)\cos 3\theta, \qquad (2.47)$$

where $\theta = \Omega_o t$. To derive this result use was made of the following trigonometric relation. Secular terms will not appear in the solution for $x_1(t)$ if the coefficient of the $\cos\theta$ term is zero, i.e.,

$$\Omega_0^2 - \left(\frac{3}{4}\right) A^3 = 0\,, \qquad (2.48)$$

and

$$\Omega_0(A) = \left(\frac{3}{4}\right)^{\frac{1}{2}} A$$

(2.49)

Under the no secular term requirement, equation (2.47) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = -\left(\frac{A^3}{4}\right)\cos 3\theta$$

(2.50)

The particular solution for this equation takes the form

$$x_1^{(p)}(t) = D\cos(3\theta)$$

Substitution of this into equation (2.50) gives

$$(-9\Omega_0^2 + \Omega_0^2)D = -\left(\frac{A^3}{4}\right)$$

and

$$D = \frac{A^3}{32\Omega_0^2} = \left(\frac{A^3}{32}\right)\left(\frac{4}{3A^2}\right) = \frac{A}{24}$$

Therefore, the full solution to equation (2.50) is

$$x_1(t) = x_1^{(h)} + x_1^{(p)} = C\cos\theta + \left(\frac{A}{24}\right)\cos 3\theta,$$

where C $\cos\theta$ is the solution to the homogeneous equation

$$\ddot{x}_1 + \Omega_0^2 x_1 = 0. \qquad (2.51)$$

Since $x_1(0) = A,$ then

$$A = C + \left(\frac{A}{24}\right)$$

or

$$C = \left(\frac{23}{24}\right)A,$$

and the full solution to equation (2.50) is

$$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos\theta + \left(\frac{1}{24}\right)\cos 3\theta\right].$$

(2.52)

If we stop the calculation at this point, then the first-approximation to the periodic solution is

$$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos\left(\sqrt{\frac{3}{4}}At\right) + \left(\frac{1}{24}\right)\cos\left(3\sqrt{\frac{3}{4}}At\right)\right].$$

(2.53)

However, to extend our calculation to the next level, $x_1(t)$ takes the form given by equation (2.44), but θ is now equal to $\Omega_1 t,$ *i.e.*,

$$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos(\Omega_1 t) + \left(\frac{1}{24}\right)\cos(3\Omega_1 t)\right]$$

$$= A\left[\left(\frac{23}{24}\right)\cos\theta + \left(\frac{1}{24}\right)\cos 3\theta\right].$$

(2.54)

Note, we denote the phase of the trigonometric expressions by θ, i.e., $\theta = \Omega_1 t$. This short-hand notation will be used for the remainder of the chapter.

The next approximation, $x_2(t)$, requires the solution to

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - x_1^3. \tag{2.55}$$

We now present the full details on how to evaluate the right-hand side of equation (2.55). These steps demonstrate what must be done for this type of calculation. In the calculations for other TNL oscillators, we will generally omit many of the explicit details contained in this section.

To begin, consider the following result

$$(a_1\cos\theta + a_2\cos 3\theta)^3 = (a_1\cos\theta)^3 + 3(a_1\cos\theta)^2(a_2\cos 3\theta) + 3(a_1\cos\theta)(a_2\cos 3\theta)^2 + (a_2\cos 3\theta)^3$$

Using $(\cos\theta_1)(\cos\theta_2) = \left(\frac{1}{2}\right)[\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)]$

and the previous expression for $(\cos\theta_1)^3$, we find

$$(a_1\cos\theta + a_2\cos 3\theta)^3 = f_1\cos\theta + f_2\cos 3\theta + f_3\cos 5\theta + f_4\cos 7\theta + f_5\cos 9\theta \tag{2.56}$$

where

$$\begin{cases} f_1 = \left(\dfrac{3}{4}\right)[a_1^3 + a_1^2 a_2 + 2a_1 a_2^2], \\ f_2 = \left(\dfrac{1}{4}\right)[a_1^3 + 6a_1^2 a_2 + 3a_2^3], \\ f_3 = \left(\dfrac{3}{4}\right)[a_1^2 a_2 + a_1 a_2^2], \\ f_4 = \left(\dfrac{3}{4}\right)a_1 a_2^2, \\ f_5 = \dfrac{a_2^3}{4} \end{cases}$$

(2.57)

For our problem, we have

$$\begin{cases} a_1 = \left(\dfrac{23}{24}\right)A \equiv \alpha A, \\ a_2 = \left(\dfrac{1}{24}\right)A \equiv \beta A. \end{cases}$$

Using these results, equation (2.55) becomes

$$\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^2 a_1 - f_1)\cos\theta + (\Omega_1^2 a_2 - f_2)\cos 3\theta - f_3 \cos 5\theta - f_4 \cos 7\theta - f_5 \cos 9\theta \,.$$

Secular terms may be eliminated in the solution for $x_2(t)$ if the coefficient of the $\cos\theta$ term is zero, i. e.,

$$\Omega_1^2 a_1 - f_1 = 0\,, \tag{2.58}$$

and

$$\Omega_1^2(A) = \frac{f_1}{a_1} = \left(\frac{3}{4}\right)[\alpha^3 + \alpha^2\beta + 2\alpha\beta^2]A^3 / \alpha A$$

$$= \left[\left(\frac{3}{4}\right)A^3\right][\alpha^2 + \alpha\beta + 2\beta^2] = \Omega_0^2(A)h(\alpha,\beta), \tag{2.59}$$

where

$$h(\alpha,\beta)=\alpha^2+\alpha\beta+2\beta^2 \tag{2.60}$$

Examination of equation (2.59) and (2.60) shows that $h(\alpha,\beta)$ provides a correction to the square of the first-order angular frequency $\Omega_0^2(A)$. Since $\alpha=\frac{23}{24}\, and\, \beta=\frac{1}{24}$, then

$$\Omega_0(A)=\sqrt{\frac{3}{4}}A=(0.866025)A, \tag{2.61}$$

$$\Omega_1(A)=(0.849326)A, \tag{2.62}$$

Let us now calculate $x_2(t)$. This function is a solution to

$$\ddot{x}_2+\Omega_1^2 x_2=(\Omega_1^2 a_2-f_2)\cos 3\theta-f_3\cos 5\theta-f_4\cos 7\theta-f_5\cos 9\theta \tag{2.63}$$

The particular solution is

$$x_2^{(p)}(t)=L_1\cos 3\theta+L_2\cos 5\theta+L_3\cos 7\theta+L_4\cos 9\theta \tag{2.64}$$

where $(L_1,\mathrm{L}_2,\mathrm{L}_3,\mathrm{L}_4)$ are constants that can be found by substituting $x_2^{(p)}$ into equation (2.63) and equating similar terms on both the left and right sides. Performing this procedure gives

$$L_1=\frac{\Omega_1^2 a_2-f_2}{(-8)\Omega_1^2}$$

$$=-\left(\frac{A}{24}\right)\left[\frac{3\beta(\alpha^2+\alpha\beta+2\beta^2)-(\alpha^3+6\alpha^2\beta+3\beta^3)}{\alpha^2+\alpha\beta+2\beta^2}\right],$$

$$L_2=\frac{f_3}{24\Omega_1^2}=\left(\frac{A}{24}\right)\left[\frac{(\alpha^2\beta+\alpha\beta^2)}{\alpha^2+\alpha\beta+2\beta^2}\right],$$

$$L_3=\frac{f_4}{48\Omega_1^2}=\left(\frac{A}{48}\right)\left[\frac{\alpha\beta^2}{\alpha^2+\alpha\beta+2\beta^2}\right],$$

$$L_4 = \frac{f_5}{80\Omega_1^2} = \left(\frac{A}{240}\right)\left[\frac{\beta^3}{\alpha^2+\alpha\beta+2\beta^2}\right],$$

In these expressions, we have replaced Ω_1^2 by the results in equation (2.59) and (2.60)

The complete solution for $x_2(t)$ is

$$x_2(t) = x_2^{(II)}(t) + x_2^{(p)} = C\cos\theta + x_2^{(p)}.$$

For t=0, we have

$$A = C + (\mathrm{L}_1 + L_2 + L_3 + L_4).$$

If we define

$L_i = A\overline{L}_i$; i=1, 2, 3, 4;

Then

$$C = 1 - (\overline{\mathrm{L}}_1 + \overline{L}_2 + \overline{L}_3 + \overline{L}_4)A,$$

and

$$x_2(t) = [1-(\overline{\mathrm{L}}_1+\overline{\mathrm{L}}_2+\overline{\mathrm{L}}_3+\overline{\mathrm{L}}_4)]A\cos\theta + A[\overline{\mathrm{L}}_1\cos 3\theta + \overline{\mathrm{L}}_2\cos 5\theta + \overline{\mathrm{L}}_3\cos 7\theta + \overline{\mathrm{L}}_4\cos 9\theta)],$$

where $\theta = \Omega_1(A)t$.

Using the numerical values for α and β, the $\overline{\mathrm{L}}$'s can be calculated; we find their values to be

$$\overline{\mathrm{L}}_1 = 0.042876301 \approx (4.29).10^{-2},$$

$$\overline{\mathrm{L}}_2 = 0.001729754 \approx (1.73).10^{-3},$$

$$\overline{\mathrm{L}}_3 = 0.000036038 \approx (3.60).10^{-5},$$

$$\overline{\mathrm{L}}_4 = 0.000000313 \approx (3.13).10^{-7}.$$

Therefore, we have for $x_2(t)$ the expression

$$x_2(t) = A[(0.955)\cos\theta + (4.29).10^{-2}\cos 3\theta + (1.73).10^{-3}\cos 5\theta + (3.60).10^{-5}\cos 7\theta + (3.13).10^{-7}\cos 9\theta]$$

Further a generalization of this work was then given by Lim and Wu (2002). Their procedure is as follows:

They assumed the equation in the form

$$\ddot{x} + f(x) = 0, x(0) = A, \dot{x}(0) = 0, \tag{2.65}$$

where A is given positive constant and $f(x)$ satisfies the condition

$$f(-x) = -f(x). \tag{2.66}$$

Adding $\omega^2 x$ on both sides of equation (2.65), we obtain

$$\ddot{x} + \omega^2 x = \omega^2 x - f(x) \equiv g(x), \tag{2.67}$$

where ω is priory unknown frequency of the periodic solution $x(t)$ being sought.

They proposed the Iterative scheme of equation (2.67)

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(x_{k-1}) + g(x_{k-1})(x_k - x_{k-1});\ k = 0,1,2,\dots, \tag{2.68}$$

where $g_x = \dfrac{\partial g}{\partial x}$ and the inputs of starting functions are

$$x_{-1}(t) = x_0(t) = A\cos(\omega t), \tag{2.69}$$

with the initial conditions

$$x_k(0) = A, \quad \dot{x}_k(0) = 0,\ k = 1, 2, 3, \dots . \tag{2.70}$$

Then substituting equation (2.69) into equation (2.68) and expanding the right hand side of equation (2.68) into the Fourier series yields

$$g[x_{k-1}(t)] + g_x[x_{k-1}(t)][x_t(t) - x_{k-1}(t)] = a_1(A,\omega)\cos\omega t + \sum_{n=2}^{N} a_{2n-1}(A,\omega)\cos[(2n-1)\omega t], \tag{2.71}$$

where the coefficients $a_{2n-1}(A,\omega)$ are known functions of A and ω, and the integer N depends upon the function $g(x)$ of the right hand side of equation (2.67) , On view of equation (2.71), the solution of equation is taken to be

$$x_{k+1}(t) = B\cos\omega t - \sum_{n=2}^{N} \frac{a_{2n-1}(A,\omega)}{[(2n-1)^2-1]\omega^2}\cos[(2n-1)\omega t], \tag{2.72}$$

where B is, tentatively, an arbitrary constant.

In equation (2.72), the particular solution is chosen such that it contains no secular terms (Mickens R E, 2010) which requires that the coefficient $a_1(A,w)$ of right-side term $\cos\omega t$ in equation (2.71) satisfy

$$a_1(A,w) = 0. \tag{2.73}$$

The equation (2.73) allows the determination of the frequency as a function A. Next, the unknown constant B will be computed by imposing the initial conditions in equation (2.70). Finally, putting these steps together gives the solution $x_{k+1}(t)$.

In 2005, this process was extended by Mickens (1987) which is used in the calculation of approximations to the periodic solutions of nonlinear oscillatory differential equations. A generalization of this work was then given by Lim and Wu (2002) and this was followed by an additional extension in Mickens. Actually, Iterative method is a technique for calculating approximations to the periodic solutions of TNL oscillator which is patented by R.E. Mickens in (1987).

Mickens (2010) has given the general procedure for calculating solutions by means of the method of Extended Iterative method as follows:

He considers the equation as

$$\ddot{x} + f(\ddot{x},\dot{x},x) = 0, \quad x(0) = A, \ \dot{x}(0) = 0, \tag{2.74}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of equation (2.74), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x},\dot{x},x) \equiv G(x,\dot{x},\ddot{x}). \tag{2.75}$$

Now, formulate the Iterative scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}) + G_x(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(x_k - x_{k-1}) \\ + G_{\dot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(\dot{x}_k - \dot{x}_{k-1}) + G_{\ddot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(\ddot{x}_k - \ddot{x}_{k-1}), \tag{2.76}$$

where

$$G_x = \frac{\partial G}{\partial x}\ ,\ G_{\dot{x}} = \frac{\partial G}{\partial \dot{x}},\ G_{\ddot{x}} = \frac{\partial G}{\partial \ddot{x}}. \tag{2.77}$$

And x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{2.78}$$

The starting function are taken to be (Mickens R E, 2005)

$$x_{-1}(t) = x_0(t) = A\cos(\Omega_0 t). \tag{2.79}$$

The right hand side of equation (2.76) is essentially the first term in a Taylor series expansion of the function $G(x_k, \dot{x}_k, \ddot{x}_k)$ at the point $(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})$ (Taylor and Mann, (1983)). To illustrate this point, note that

$$x_k = x_{k-1} + (x_k - x_{k-1}), \tag{2.80}$$

and for some function $G(x)$, we have

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] = G(x_{k-1}) + G_x(x_k - x_{k-1}) + \dots. \tag{2.81}$$

An alternative, but very insightful, modification of above scheme was proposed by Hu H (2006)(a),(b). He used the following equation in place of equation (2.80)

$$x_k = x_0 + (x_k - x_o). \tag{2.82}$$

Then, equation (2.81) is changed to

$$G(x_k) = G[x_0 + (x_k - x_0)] = G(x_0) + G_x(x_k - x_0) + \dots, \tag{2.83}$$

and the corresponding modification to equation (2.76) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_0,\ \dot{x}_0,\ \ddot{x}_0); + G_x(x_0,\ \dot{x}_0,\ \ddot{x}_0)(x_k - x_0) \\ + G_{\dot{x}}(x_0,\ \dot{x}_0,\ \ddot{x}_0)(\dot{x}_k - \dot{x}_0) + G_{\ddot{x}}(x_0,\ \dot{x}_0,\ \ddot{x}_0)(\ddot{x}_k - \ddot{x}_0). \tag{2.84}$$

This scheme is computationally easier to work with, for $k \geq 2$, than the one given in equation (2.75). The essential idea is that if $x_0(t)$ is a good approximation, then the expansion should take place at $x = x_0$. Also, as pointed out by Hu H, 2006(b), the $x_0(t)$ in $(x_k - x_o)$ is not the same for all k. In particular, $x_0(t)$ in $(x_1 - x_o)$ is the function $A\cos(\Omega_1 t)$, while the $x_0(t)$ in $(x_2 - x_o)$ is the function $A\cos(\Omega_2 t)$.

Here we present an Example performed by Mickens with the Extended Iterative method (Mickens R E, 2010):

Let us consider the nonlinear oscillator given by

$$\ddot{x} + x^{-1} = 0. \tag{2.85}$$

The TNL oscillator equation (2.85) has several possible Iterative schemes. We use the one derived from the relation

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x(\ddot{x})^2 = G(x, \ddot{x}, \Omega^2), \tag{2.86}$$

that is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \left[\Omega_k^2 x_0 - x_0(\ddot{x}_0)^2\right] + \left[\Omega_k^2 - (\ddot{x}_0)^2\right](x_k - x_0) - 2x_0\ddot{x}_1(\ddot{x}_k - \ddot{x}_0). \tag{2.87}$$

To obtain this relation the following formula was used for the Extended Iterative scheme

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_0, \ddot{x}_0, \Omega_k^2) + G_x(x_0, \ddot{x}_0, \Omega_k^2)(x_k - x_0) + G_{\ddot{x}}(x_0, \ddot{x}_0, \Omega_k^2)(\ddot{x}_k - \ddot{x}_0). \tag{2.88}$$

For k = 1, we have

$$\ddot{x}_2 + \Omega_1^2 x_2 = 2x_0(\ddot{x}_0)^2 + \left[\Omega_1^2 - (\ddot{x}_0)^2\right]x_1 - 2x_0\ddot{x}_0\ddot{x}_1, \tag{2.89}$$

with

$$\begin{cases} x_0(t) = A\cos\theta, \\ x_1(t) = A\left[\alpha\cos\theta + \beta\cos 3\theta\right], \\ \theta = \Omega_1 t,\ \ \alpha = \dfrac{23}{24},\ \ \beta = \dfrac{1}{24}. \end{cases} \tag{2.90}$$

In the from this equation (2.85)

Then the particular solution, $x_1^{(p)}(t)$, is

$$x_1^{(p)}(t) = \left(\frac{A^3\Omega_0^2}{32}\right)\cos 3\theta = \left(\frac{A}{24}\right)\cos 3\theta. \tag{2.91}$$

Therefore, the full solution is

$$x_1(t) = C\cos\theta + \left(\frac{A}{24}\right)\cos 3\theta. \tag{2.92}$$

Using $x_1(0) = A,$ then $C = 23/24$ and

$$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos\theta + \left(\frac{1}{24}\right)\cos 3\theta\right]. \tag{2.93}$$

(See equation (2.93) for $x_1(t)$). Substitution of the items in equation (2.90) into the right-hand side of $\Omega_1(A) = \dfrac{1.189699}{A}$. Equation (2.89) gives, after some algebraic and trigonometric simplification, the result

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = (\Omega_1^2 A)&\left[\alpha - (3-7\beta)\left(\frac{\Omega_1^2 A^4}{4}\right)\right]\cos\theta \\ &-\left(\frac{A\Omega_1^2}{4}\right)[(1+35\beta)\Omega_1^2 A^2 - 4\beta]\cos 3\theta - \left(\frac{19\beta}{4}\right)(\Omega_1^4 A^3)\cos 5\theta.\end{aligned} \tag{2.94}$$

Setting the coefficient of $\cos\theta$ to zero and solving for Ω_1^2 gives

$$\begin{cases}\Omega_1^2(A) = \left[\left(\dfrac{4}{3}\right)\dfrac{1}{A^2}\right]\left(\dfrac{69}{65}\right) = \Omega_0^2(A)\left[\dfrac{69}{65}\right] \\ \Omega_1(A) = \dfrac{1.189699}{A}.\end{cases} \tag{2.95}$$

Comparing $\Omega_1(A)$ with the exact value, $\Omega_{exact}(A)$, we find the following percentage error

$$\left|\frac{\Omega_{exact} - \Omega_1}{\Omega_{exact}}\right| \times 100 = 5.1\%\ error.$$

Note that using the direct Iterative scheme, we found

$$\begin{cases} \Omega_0(A) = \dfrac{1.1547}{A} \;\; (7.9\% \; error), \\ \Omega_1(A) = \dfrac{1.0175}{A} \;\; (18.1\% \; error). \end{cases}$$

Therefore, the Extended Iterative procedure provides a better estimate of the angular frequency.

Replacing $\Omega_1^2 A^2$ in Eq. (2.94), by the expression of equation (2.95), we obtain

$$\ddot{x}_2 + \Omega_1^2 x_2 = -\left(\frac{A\Omega_1^2}{4}\right)\left(\frac{1292}{390}\right)\cos\theta - \left(\frac{A\Omega_1^2}{4}\right)\left(\frac{437}{390}\right)\cos 5\theta. \tag{2.96}$$

The corresponding particular solution takes the form

$$x_2^{(p)}(t) = D_1 \cos 3\theta + D_2 \cos 7\theta. \tag{2.97}$$

Substituting this into equation (2.96) and equating the coefficients, respectively, of the $\cos 3\theta$ and $\cos 7\theta$ terms, allows the calculation of D_1 and D_2; they are

$$\begin{cases} D_1 = \left(\dfrac{3876}{37440}\right) A, \\ D_2 = \left(\dfrac{437}{37440}\right) A. \end{cases} \tag{2.98}$$

Since the full solution for $x_2(t)$ is

$$x_2(t) = C \cos\theta + x_1^{(p)}(t), \tag{2.99}$$

with $x_2(0) = A$, it follows that

$$C = A - D_1 - D_2 = \left(\frac{33127}{37440}\right) A, \tag{2.100}$$

and

$$\begin{cases} x_2(t) = A\left[\left(\dfrac{33127}{37440}\right)\cos\theta + \left(\dfrac{3876}{37440}\right)\cos 3\theta + \left(\dfrac{437}{37440}\right)\cos 5\theta\right], \\ \theta = \Omega_1(t)t = \left[\dfrac{92}{65}\right]^{1/2}\left(\dfrac{1}{A}\right). \end{cases} \tag{2.101}$$

Inspection of $x_2(t)$ indicates that the coefficients of the harmonics satisfy the ratios

$$\begin{cases} \dfrac{a_1}{a_0} = \dfrac{3876}{33127} \approx 0.117, \\ \dfrac{a_2}{a_1} = \dfrac{437}{3876} \approx 0.113. \end{cases} \tag{2.102}$$

Now-a-day's Iterative method is used widely by Lim and Wu (2002), Hu and Tang (2006)(b), Chen and Liu (2008), Haque *et al.*, 2013; Haque B M I, 2013; Haque B M I, 2014; Haque *et al.*, 2014; Haque and Hossain, 2016; Haque *et al.*, 2016(a); Haque *et al.*, 2016(b); Haque *et al.*, 2016(c); Haque *et al.*, 2017) etc. which is valid for small together with large amplitude of oscillation to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Mickens (1987) provided a general basis for Iterative methods as they are currently used in the calculation of approximations to the periodic solutions of nonlinear oscillatory differential equations.

In Haque's Iteration method the problem is solved by the following way-

Let us consider the Oscillator

$$\ddot{x} + x^{-1} = 0. \tag{2.103}$$

Adding $\Omega^2 x$ on both sides of equation (3.6), we get

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - x^{-1} \tag{2.104}$$

According to equation (2.40), the Iterative scheme of equation (2.104) will be

$$\ddot{x}_{k+1} + \Omega_k^2\, x_{k+1} = \Omega_k^2\, x_k - x_k^{-1}. \tag{2.105}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained from the solution of (putting $k = 0$ in equation (2.105) and utilizing equation (2.41)

$$\ddot{x}_1 + \Omega_0^2\, x_1 = \Omega_0^2\, A\cos\theta - (A\cos\theta)^{-1}. \tag{2.106}$$

Now expanding $(\cos\theta)^{-1}$ in a Fourier Cosine series in interval $[0, \pi]$, the equation (2.104) reduces to

$$\ddot{x}_1 + \Omega_0^2\, x_1 = \Omega_0^2\, A\cos\theta - \frac{2}{A}\sum_{n=1}^{\infty}(-1)^{n-1}\cos(2n-1)\theta. \tag{2.107}$$

To check secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (2.107), and we obtain

$$\Omega^2{}_0 = \frac{2}{A^2},\ \Omega_0 = \frac{1.414}{A}. \tag{2.108}$$

Then solving equation (2.107) and satisfying the initial condition (according to equation (2.42)), we obtain

$$x_1(t) = A((1+\frac{1}{4}(-1+2\ln 2))\cos\theta - \sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta). \tag{2.109}$$

This is the second approximation of equation (2.103) and the related Ω_1 is to be determined. The second approximation $x_2(t)$ and the value of Ω_1 are obtained from the solution of

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - x_1^{-1}. \tag{2.110}$$

Substituting $x_1(t)$ from equation (2.109) into the right-hand side of equation (2.110), we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = A\Omega_1^2((1+(-1+2\ln 2)/4)\cos\theta - \sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta) \\ -\frac{1}{A}\sum_{n=1}^{\infty}(-1)^{n-1}a_{2n-1}\cos(2n-1)\theta,\end{aligned} \tag{2.111}$$

where

$$a_1 = 1.599611, a_3 = 0.983636, a_5 = 1.102235, a_7 = 1.079400, a_9 = 1.083797, \ldots. \tag{2.112}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (2.111). Thus we have

$$\Omega^2{}_1 = \frac{1.599611}{A^2(1+(-1+2\ln 2)/4)},\ \Omega_1 = \frac{1.208}{A}. \tag{2.113}$$

Then equation (2.111) becomes,

$$\ddot{x}_2 + \Omega_1^2 x_2 = -A\Omega_1{}^2\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta + \frac{1}{A}\sum_{n=2}^{\infty}(-1)^n a_{2n-1}\cos(2n-1)\theta. \tag{2.114}$$

The equation (2.114) approximately can be written as,

$$\ddot{x}_2 + \Omega_1^2 x_2 = -A\Omega_1{}^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{4(n-1)n} \cos(2n-1)\theta + \frac{1.1}{A} \sum_{n=2}^{\infty} (-1)^n a_{2n-1} \cos(2n-1)\theta \,. \tag{2.115}$$

Then solving equation (2.115) and satisfying the initial condition, we obtain the second approximation,

$$\begin{aligned} x_2(t) = {} & A((1-(3-4\ln 2)/16 + 1.1(-1+2\ln 2)/(4z))\cos\theta A \\ & + \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{(4(n-1)n)^2} + \frac{1.1(-1)^{n-1}}{4(n-1)n\Omega^2{}_1} \right) \cos(2n-1)\theta, \end{aligned} \tag{2.116}$$

where

$$z = \frac{8}{(1+(-1+2\ln 2)/4)\sqrt{(3+\ln 2)(4+\ln 16)}} \,. \tag{2.117}$$

The third approximation x_3 and the value of Ω_2 are obtained from the solution of

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - x_2^{-1} \,. \tag{2.118}$$

Substituting $x_2(t)$ from equation (2.116) into the right-hand side of equation (2.118) and utilizing the same method, we obtain

$$\ddot{x}_3 + \Omega_2^2 x_3 = \sum_{n=2}^{\infty} \left(A\Omega^2{}_2 \left(\frac{(-1)^n}{(4(n-1)n)^2} + \frac{1.1(-1)^{n-1}}{4\Omega^2{}_1(n-1)n} \right) + (-1)^n \frac{1.26}{A} \right) \cos(2n-1)\theta \,, \tag{2.119}$$

where

$$\Omega_2^2 = 1.693744 / A^2(1-(3-4\ln 2)/16 + 1.1(-1+2\ln 2)/(4z)) \,,\ \Omega_2 = \frac{1.265}{A}. \tag{2.120}$$

Then solving equation (2.118) and satisfying the initial condition, we obtain

$$x_3(t) = A\left(1.0672\cos\theta - \sum_{n=2}^{\infty} \left(\frac{1}{((n-1)n)^3} - \frac{1.1}{z((n-1)n)^2} + \frac{1.26}{z_1(n-1)n} \right) \cos\theta(2\cos 2\theta - 1) \right), \tag{2.121}$$

where

$$z_1 = 1.693744/(1-(3-4\ln 2)/16 + 1.1(-1+2\ln 2)/4z) \,. \tag{2.122}$$

Therefore, $\Omega_0, \Omega_1, \Omega_2, \ldots$, respectively obtained by equation (2.108), (2.113), (2.120),…., represent the approximation of frequencies of oscillator (2.103).

Further Mickens used the iterative technique (Mickens R E, 2006) to calculate a higher-order approximation to the periodic solutions of a conservative oscillator for which the elastic force term is proportional to $x^{\frac{1}{3}}$. Hu (2006(a), (b)) applied the modified iteration technique of (Mickens R E, 2005) to find approximate of nonlinear oscillators with fractional powers and quadratic nonlinear oscillator respectively. Recently, Zheng et al. (2013) has applied Mickens extended iteration method and direct iteration method to determine approximate periodic solutions of a class of nonlinear jerk equations.

CHAPTER III

Extended Iterative Method

3.1 Introduction

The main purpose of this thesis is to develop the Extended Iterative technique for the determination of approximate solution and angular frequency of “the quadratic nonlinear oscillator”. The results will be compared with existing results obtained by various researchers and it is expected that the obtain results by this techniques would be similar and sometimes better results than other existing procedures.

3.2 The method

An Extended Iterative method will be used to obtain analytical solution of the quadratic nonlinear oscillator. The procedure may be briefly described as follows.

A nonlinear oscillator will be modeled by

$$\ddot{x}+f(x)=0,\ x(0)=A,\ \dot{x}(0)=0, \tag{3.1}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of equation (3.1), we obtain

$$\ddot{x}+\Omega^2 x=\Omega^2 x-f(x)\equiv G(\mathrm{x},\Omega). \tag{3.2}$$

The Extended Iterative scheme is

$$\ddot{x}_{k+1}+\Omega_k^2\, x_{k+1}=G(x_{k-1},\Omega)+G_x(x_{k-1},\Omega)(x_k-x_{k-1});\ \ k=1,2,\ldots \tag{3.3}$$

where $G_x=\dfrac{\partial G}{\partial x}$.

The right hand side of equation (3.3) is essentially the first term in a Taylor series expansion of the function $G(x_k,\dot{x}_k)$ at the point $(x_{k-1},\dot{x}_{k-1})$ (Taylor and Mann, 1983)

We have the direct iteration scheme of equation (3.2) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \Omega_k); \quad k = 0,1,2,... \tag{3.4}$$

and x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A. \tag{3.5}$$

The initial guess are taken to be (Lim and Wu, 2002)

$$x_0(t) = A\cos(\Omega_0 t). \tag{3.6}$$

The above procedure gives the sequence of solutions $x_1(t),\ x_2(t), x_3(t), \cdots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second (Mickens R E, 1987)

3.3 Solution Procedure

Let us consider the nonlinear inverse oscillator

$$\ddot{x} + x^2 = 0 \tag{3.7}$$

Adding $\Omega^2 x$ on both sides of equation (3.7), we get

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^2 = G(x, \Omega), \tag{3.8}$$

where $G(x,\Omega) = \Omega^2 x - x^2$, $G_x(x,\Omega) = \Omega^2 - 2x.$

According to equation (3.4), the direct Iterative scheme of equation (3.8) is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^2 \tag{3.9}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting $k = 0$ in equation (3.9) and using equation (3.6) we get

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 x_0 - x_0^2 \ , \tag{3.10}$$

where $x_0(t) = A\cos(\Omega_0 t) = A\cos\theta$ and $\theta = \Omega_0 t$

Now substituting $x_0(t)$ and expanding the right-hand-side in a Fourier cosine series, then equation (3.10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - (A\cos\theta)^2$$

$$\begin{aligned}\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - A^2 (0.848826\cos\theta + 0.169765\cos 3\theta \\ - 0.024252\cos 5\theta + 0.008084\cos 7\theta - 0.003675\cos 9\theta \\ + 0.001979\cos 11\theta)\end{aligned}$$

$$\begin{aligned}= (\Omega_0^2 - 0.848826A)A\cos\theta - A^2 (0.169765\cos 3\theta + 0.024252\cos 5\theta \\ - 0.008084\cos 7\theta + 0.003675\cos 9\theta - 0.001979\cos 11\theta).\end{aligned} \tag{3.11}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (3.11). Thus we have

$$\Omega_0^2 A - 0.848826A^2 = 0,\ \Omega_0{}^2 = \frac{0.848826A^2}{A},\ \Omega_0 = 0.921318\sqrt{A}. \tag{3.12}$$

This is the first approximate frequency of the oscillator. Note that $\Omega_{exact}(A) = 0.914681\sqrt{A}$. After simplification the equation (3.11) reduces to

$$\begin{aligned}\ddot{x}_1 + \Omega_0^2 x_1 = A^2 (-0.169765\cos 3\theta + 0.024252\cos 5\theta - 0.008084\cos 7\theta \\ + 0.003675\cos 9\theta - 0.001979\cos 11\theta).\end{aligned} \tag{3.13}$$

The particular solution, $x_1^{(p)}(t)$ is

$$\begin{aligned}x_1^{(p)}(t) = \frac{-0.169765A^2}{-9\Omega_0^2 + \Omega_0^2}\cos 3\theta + \frac{0.024252A^2}{-25\Omega_0^2 + \Omega_0^2}\cos 5\theta + \frac{-0.008084A^2}{-49\Omega_0^2 + \Omega_0^2}\cos 7\theta \\ + \frac{0.003675A^2}{-81\Omega_0^2 + \Omega_0^2}\cos 9\theta + \frac{0.001979A^2}{-121\Omega_0^2 + \Omega_0^2}\cos 11\theta\end{aligned}$$

$$\begin{aligned}= \frac{A^2}{\Omega_0^2}\left(\frac{0.169765}{8}\cos 3\theta - \frac{0.024252}{24}\cos 5\theta + \frac{0.008084}{48}\cos 7\theta \right. \\ \left. - \frac{0.003675}{80}\cos 9\theta - \frac{0.001979}{120}\cos 11\theta\right) \\ = A(0.025\ \cos 3\theta - 0.001190\ \cos 5\theta + 0.000199\ \cos 7\theta \\ - 0.000054\ \cos 9\theta - 0.000019\ \cos 11\theta)\end{aligned} \tag{3.14}$$

Therefore, the complete solution is

$$x_1(t) = B_1 \cos\theta + 0.025\, A \cos 3\theta - 0.001190\, A \cos 5\theta$$
$$+ 0.000199\, A \cos 7\theta - 0.000054\, A \cos 9\theta \tag{3.15}$$
$$- 0.000019\, A \cos 11\theta.$$

Using $x_1(0) = A$, we have $B_1 = 0.976066\, A$. Then we obtain

$$x_1(t) = \mathrm{A}(0.976066\ \cos\theta + 0.025\ \cos 3\theta$$
$$-\ 0.00119048\ \cos 5\theta + 0.000198413\ \cos 7\theta \tag{3.16}$$
$$-\ 0.0000541126\ \cos 9\theta - 0.000019425\ \cos 11\theta).$$

This is the first approximate solution of the oscillator.

According to equation (3.3), the Extended Iterative scheme of equation (3.8) is

$$\ddot{x}_{k+1} + \Omega_k^2\, x_{k+1} = (\Omega^2\, x_k - x_k^2) + (\Omega^2 - 2x_k)(x_k - x_{k-1}).$$
$$= x_k^2 + \Omega^2 x_{k-1} - 2x_k x_{k-1} \tag{3.17}$$

The second approximation $x_2(t)$ and the frequency Ω_1 will be obtained by putting $k = 1$ in equation (3.17) and using equation (3.7) we get (3.18)

$$\ddot{x}_2 + \Omega_1^2\, x_2 = x_1^2 + \Omega^2 x_0 - 2x_1 x_0\,, \tag{3.18}$$

Where $x_0(t)$ and $x_1(t)$ are given by the equations (3.6) and (3.16).

Now substituting $x_0(t)$ and $x_1(t)$ are expanding the right- hand side in a Fourier cosine series, then equation (3.17) reduces to

$$\ddot{x}_2 + \Omega_1^2\, x_2 = (\Omega_1^2\, 0.976066A - 0.816744A^2)\cos\theta + (\Omega_1^2\, 0.025000A$$
$$-0.193883A^2)\cos 3\theta + (-\Omega_1^2\, 0.001190A + 0.014431A^2)\cos 5\theta$$
$$+(\Omega_1^2\, 0.000198A - 0.005635A^2)\cos 7\theta + (-\Omega_1^2\, 0.000054A$$
$$+0.002713A^2)\cos 9\theta + (-\Omega_1^2\, 0.000019A - 0.001441A^2)\cos 11\theta$$
$$+0.000911A^2 \cos 13\theta \tag{3.19}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (3.19). Thus we have

$$\Omega_1^2\, 0.976066A - 0.816744A^2 = 0,\ \ \Omega_1 = 0.914752\sqrt{A}. \tag{3.20}$$

The particular solution, $x_2^{(p)}(t)$ is

$$x_2^{(p)}(t) = \frac{1}{-9\Omega_1^2+\Omega_1^2}(\Omega_1^2\,0.025000A - 0.193883A^2)\cos 3\theta + \frac{1}{-25\Omega_1^2+\Omega_1^2}(-\Omega_1^2\,0.001190A + 0.014431A^2)\cos 5\theta + \frac{1}{-48\Omega_1^2+\Omega_1^2}(\Omega_1^2\,0.000198A - 0.005635A^2)\cos 7\theta + \frac{1}{-81\Omega_1^2+\Omega_1^2}(-\Omega_1^2\,0.000054A + 0.002712A^2)\cos 9\theta + \frac{1}{-121\Omega_1^2+\Omega_1^2}(-\Omega_1^2\,0.000019A - 0.001441A^2)\cos 11\theta + \frac{1}{-169\Omega_1^2+\Omega_1^2}(0.000911A^2)\cos 13\theta$$

$$= -\frac{0.025000A}{8}\cos 3\theta + \frac{0.193883A^2}{8\Omega_1^2}\cos 3\theta - \frac{0.001190A}{24}\cos 5\theta - \frac{0.014431A^2}{24\Omega_1^2}\cos 5\theta - \frac{0.000198A}{48}\cos 7\theta + \frac{0.005635A^2}{48\Omega_1^2}\cos 7\theta + \frac{0.000054A}{80}\cos 9\theta - \frac{0.002712A^2}{80\Omega_1^2}\cos 9\theta + \frac{0.000019A}{120}\cos 11\theta + \frac{0.001441A^2}{120\Omega_1^2}\cos 11\theta - \frac{0.000911A^2}{168\Omega_1^2}\cos 13\theta$$

$$= 0.025838A\cos 3\theta - 0.000669A\cos 5\theta + 0.000136A\cos 7\theta - 0.000040A\cos 9\theta + 0.000015A\cos 11\theta - 6.48356\times 10^{-}6\,A\cos 13\theta \qquad (3.21)$$

Therefore, the complete solution is

$$x_2(t) = B_2\cos\theta + 0.025838A\cos 3\theta - 0.000669A\cos 5\theta + 0.000136A\cos 7\theta - 0.000040A\cos 9\theta + 0.000015A\cos 11\theta - 6.48356\times 10^{-6}A\cos 13\theta \qquad (3.22)$$

Using $x_2(0) = A$, we have $B_2 = 0.974727$ A. Then we obtain

$$x_2(t) = 0.974727A\mathit{Cos}\theta + 0.0258379A\cos 3\theta - 0.000669A\cos 5\theta + 0.000136A\cos 7\theta - 0.000040A\cos 9\theta + 0.000015A\cos 11\theta - 6.48356\times 10^{-6}A\cos 13\theta \qquad (3.23)$$

This is the second approximate solution of the oscillator.

Proceeding to the third level of Iterative, $x_3(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2 x_3 = x_2^2 + \Omega^2 x_1 - 2x_2 x_1, \tag{3.24}$$

where $x_0(t) = A\cos(\Omega_0 t) = A\cos\theta$ and

$$x_2(t) = A(0.974727Cos\theta + 0.0258379cos3\theta - 0.000669cos5\theta + 0.000136cos7\theta$$
$$- 0.000040cos9\theta + 0.000015cos11\theta - 6.48356\times10^{-6}cos13\theta)$$

Now substituting $x_1(t)$ and $x_2(t)$ and expanding the right hand side in a Fourier cosine series, then equation (3.24) reduces to

$$\begin{aligned}\ddot{x}_3 + \Omega_2^2 x_3 = &(-\Omega_2^2 0.974727A - 0.815477A^2)\cos\theta + (\Omega_2^2 0.025838A \\ &- 0.193801A^2)\cos 3\theta + (-\Omega_2^2 0.000669A + 0.013352A^2)\cos 5\theta \\ &+ (\Omega_2^2 0.000136A - 0.0058997A^2)\cos 7\theta + (-\Omega_2^2 0.0000399A \\ &+ 0.002779A^2)\cos 9\theta + (\Omega_2^2 0.000015A - 0.001523A^2)\cos 11\theta \\ &+ (-\Omega_2^2 0.000006A + 0.000924A^2)\cos 13\theta - 0.000597A^2\cos 15\theta \\ &+ 0.000412A^2\cos 17\theta - 0.000296A^2\cos 19\theta + 0.000219A^2\cos 21\theta \\ &+ 0.000167A^2\cos 23\theta + 0.0001298A^2\cos 25\theta - 0.000103A^2\cos 27\theta\end{aligned} \tag{3.25}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of equation (3.25). Thus we have

$$-\Omega_2^2 0.974727A - 0.815477A^2 = 0,\ \ \Omega_2 = 0.91467\sqrt{A}. \tag{3.26}$$

The particular solution, $x_3^{(p)}(t)$ is

$$x_3^{(p)}(t) = -\frac{1}{8\Omega_2^{\ 2}}(\Omega_2^{\ 2}\,0.025838A - 0.193801A^2)\cos 3\theta - \frac{1}{24\Omega_2^{\ 2}}$$
$$(-\Omega_2^{\ 2}\,0.000669A + 0.013352A^2)\cos 5\theta - \frac{1}{48\Omega_2^{\ 2}}(\Omega_2^{\ 2}\,0.000136A$$
$$-0.0058998A^2)\cos 7\theta - \frac{1}{80\Omega_2^{\ 2}}(-\Omega_2^{\ 2}\,0.0000399A + 0.002779A^2)$$
$$\cos 9\theta - \frac{1}{120\Omega_2^{\ 2}}(\Omega_2^{\ 2}\,0.000015A - 0.001523A^2)\cos 11\theta - \frac{1}{168\Omega_2^{\ 2}}$$
$$(-\Omega_2^{\ 2}\,0.000006A + 0.000924A^2)\cos 13\theta - \frac{1}{224\Omega_2^{\ 2}}(-0.000597A^2)$$
$$\cos 15\theta - \frac{1}{288\Omega_2^{\ 2}}(0.000412A^2)\cos 17\theta - \frac{1}{360\Omega_2^{\ 2}}(-0.000296A^2)$$
$$\cos 19\theta - \frac{1}{440\Omega_2^{\ 2}}(0.000219A^2)\cos 21\theta - \frac{1}{528\Omega_2^{\ 2}}(0.000167A^2)$$
$$\cos 23\theta - \frac{1}{624\Omega_2^{\ 2}}(0.0001298A^2)\cos 25\theta - \frac{1}{728\Omega_2^{\ 2}}(-0.000103A^2)\cos 27\theta$$

$$\begin{aligned}
&= 0.025726A cos 3\theta - 0.000637A cos 5\theta + 0.00014A cos 7\theta \\
&\quad - 0.000041A cos 9\theta + 0.000015A cos 11\theta - 6.53694\times10^{-6} A cos 13\theta \\
&\quad + 3.18798\times10^{-6} A cos 15\theta - 1.71121\times10^{-6} A cos 17\theta \\
&\quad + 9.81306\times10^{-7} A cos 19\theta - 5.94814\times10^{-7} A cos 21\theta \\
&\quad - 3.77331\times10^{-7} A cos 23\theta - 2.4863\times10^{-7} A cos 25\theta \\
&\quad + 1.69177\times10^{-7} A cos 27\theta
\end{aligned} \tag{3.27}$$

Therefore, the complete solution is

$$
\begin{aligned}
x_3(t) = {} & B_3\, cos\theta + 0.0257261 A cos3\theta - 0.000637 A cos5\theta \\
& + 0.000144 \text{ A } cos7\theta - 0.0000410252 \text{ A } cos9\theta \\
& + 0.0000150452 A\, cos11\theta - 6.53694\times10^{-6} A\, cos13\theta \\
& + 3.18798\times10^{-6} A\, cos15\theta - 1.71121\times10^{-6} A cos17\theta \\
& + 9.81306\times10^{-7} A cos19\theta - 5.94814\times10^{-7} A cos21\theta \\
& - 3.77331\times10^{-7} A cos23\theta - 2.4863\times10^{-7} A cos25\theta \\
& + 1.69177\times10^{-7} A cos27\theta
\end{aligned}
\tag{3.28}
$$

Using $x_3(0) = A$, we have $B_3 = 0.974798$ A. Then we obtain

$$
\begin{aligned}
x_3(t) = {} & 0.974798 \text{ A } \cos\theta + 0.025726 \text{ A } \cos3\theta - 0.000637 \text{ A } cos5\theta \\
& + 0.000144 \text{ A} cos7\theta - 0.000041 \text{ A } cos9\theta + 0.000015 \text{ A } cos11\theta \\
& - 6.53694\times10^{-6} \text{ A} cos13\theta + 3.18798\times10^{-6} \text{ A} cos15\theta \\
& - 1.71121\times10^{-6} \text{ A} cos17\theta + 9.81306\times10^{-7} \text{ A} cos19\theta \\
& - 5.94814\times10^{-7} \text{A} cos21\theta - 3.77331\times10^{-7} \text{A} cos23\theta \\
& - 2.4863\times10^{-7} \text{A} cos25\theta + 1.69177\times10^{-7} \text{A} cos27\theta
\end{aligned}
\tag{3.29}
$$

This is the third approximate solution of the oscillator.

CHAPTER IV

Results and Discussion

In this chapter, the results obtained by all of the methods have been compared to their periodic or oscillatory solutions. Accordingly, one measure of the accuracy or quality of a given method is the difference between the exact value of the angular frequency and that determined using the approximation procedure.

4.1 Results

An Iterative approach is presented to obtain approximate solution of the "quadratic nonlinear oscillators". The present technique is very simple for solving algebraic equations analytically and the approach is different from the existing other approach for taking truncated Fourier series. This process significantly improves the results.

Here calculated the first, second and third approximate frequencies Ω_0, Ω_1 and Ω_2 have been calculated and all the results are given in the following Table-4.1.

To compare the approximate frequencies we have also given the existing results determined by Mickens and Ramadhani (1992), Belendez et al.,(2009)(c),Hosen M A (2013)and Haque and Hossain (2016)(a), shownin the Table-4.2. Fortunately, this current method gives significantly better result than other formula.

To show the accuracy, it is calculated the percentage of errors by the following definition:

$$Error = \left| \frac{\Omega_e - \Omega_k}{\Omega_e} \right| \times 100\%$$

Where $\Omega_k \ (k = 0,1,2,...)$ represents the approximate frequencies obtained by the present method and Ω_e represents the corresponding exact frequency of the oscillator.

Table-4.1

Adopted approximate frequencies of $\ddot{x} + x^2 = 0$.

Exact frequency $\Omega_e = 0.914681\sqrt{A}$					
Amplitude A					
First approximate frequencies, Ω_0	$0.921318\sqrt{A}$	Second approximate frequencies, Ω_1	$0.914752\sqrt{A}$	Third approximate frequencies, Ω_2	$0.91467\sqrt{A}$
Error (%)	0.73	Error (%)	0.0078	Error (%)	0.0012

Table-4.2

Comparison of the approximate frequencies with exact frequency Ω_e of $\ddot{x}+x^2=0$.

Exact frequency $\Omega_e = 0.914681\sqrt{A}$			
Amplitude A	First approximate frequencies, Ω_0 & Error (%)	Second approximate frequencies, Ω_1 & Error (%)	Third approximate frequencies, Ω_2 & Error (%)
Mickens and Ramadhani (1992)	$0.921318\sqrt{A}$ 0.73	$0.914044\sqrt{A}$ 0.70	-----
Belendez *et al.* (2009)(c)	$0.921318\sqrt{A}$ 0.73	$0.914274\sqrt{A}$ 0.045	$0.914711\sqrt{A}$ 0.0032
Hosen M A (2013)	$0.921318\sqrt{A}$ 0.73	$0.914427\sqrt{A}$ 0.028	$0.914733\sqrt{A}$ 0.0056
Haque and Hossain (2016)(a)	$0.921318\sqrt{A}$ 0.73	$0.915114\sqrt{A}$ 0.047	$0.914705\sqrt{A}$ 0.0026
Adopted method	$0.921318\sqrt{A}$ 0.73	$0.914752\sqrt{A}$ 0.0078	$0.91467\sqrt{A}$ 0.0012

4.2 Convergence and Consistency Analysis

We know the basic idea of Iterative methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \qquad \text{or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here x_e is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution yield the less error in each Iterative step compared to the previous Iterative step and finally $|\Omega_2 - \Omega_e| = |0.91467 - 0.914681| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An Iterative method of the form represented by equation (3.4) with initial guess given in equation (3.5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \quad \text{or,} \quad \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0$$

In the present analysis we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0 \text{, as } |\Omega_2 - \Omega_e| = 0.$$

Thus the consistency of the method is achieved.

4.3 Discussion

It is noted that Mickens and Ramadhani (1992) found only second approximate frequencies by Harmonic Balance method. Belendez et al., (2009) (c) found up to third approximate frequencies by using modified He's Homotopy Perturbation method. Again Hosen M. A., (2013) found up to third approximate frequencies by using modified Harmonic Balance method, Haque and Hossain, (2016) (a) found up to fourth approximate frequencies by Iteration method.

In our study, it is seen that the third-order approximate frequency obtained by Adopted method is almost same with exact frequency. It is found that, in most of the cases our solution gives significantly better result than other existing results. The advantages of this method include its simplicity and computational efficiency.

CHAPTER V

Conclusions and Recommendations

In this final chapter, some concluding remarks have been included. Some essential recommendations about Extended Iterative method have also been presented.

5.1 Conclusions

The basic groundwork behind Iterative methods is to re-express the original nonlinear differential equation that includes with a vast sequence of equations, each of which can be solved, and such that at a particular stage of the calculation, knowledge of the solutions of the previous members of the sequence is required to solve the differential equation at that stage. In this thesis we used a simple but effective modification of the Extended Iterative method to investigate nonlinear differential equations. The results have improved when we truncated eleven terms to calculate the first approximate solution, thirteen terms to calculate the second approximate solution and twenty seven terms to calculate the third approximate solution. This technique can be used as paradigms for many others applications in searching for periodic solution of other nonlinear oscillators. The obtained results show that the modification of the Extended Iterative method is more accurate than other methods and is valid for large region.

5.2 Recommendations

In the final analysis, the validity and value of a particular method and the solutions that it produces depend heavily on what we intend to do with the results obtained from the calculations. However, the following issues are of prime importance:

i. A given truly nonlinear (TNL) oscillator equation may have more than one possible Iterative scheme. At present, there are no a priori meta-principles which place limitations on the construction of Iterative schemes.
ii. For level $k \geq 2$ calculations, the work required to determine the angular frequency and associated periodic solution may become algebraically intensive.
iii. The Extended Iterative method generally is easier to apply, for better result, in comparison with similar direct Iterative techniques.
iv. In principle, Iterative methods may be generalized to higher-order differential equations.

We can get desirable solution or angular frequency from a truly nonlinear oscillator by the proper use of the term of the Fourier series. In each of the Iterative scheme, right choice of truncation is most important.

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