A STUDY ON SOME DISCRETE TRANSFORMS OF ENGINEERING SCIENCES

BY

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Declaration

This is declared that the thesis entitled “A study on some discrete transforms of engineering sciences” has been carried out by Barna Shil in the Department of mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any other degree or diploma.

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Approval

This is to certify that the thesis work submitted by Barna Shil, Roll number-1451561 entitled "**A study on some discrete transforms of engineering sciences**" has been approved by the Board of examiners for the partial fulfillment of the requirements for the degree of Master of Science in Mathematics Khulna University of Engineering & Technology, Khulna, Bangladesh in May, 2016.

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All praise of Almighty God who has created us and given me a greatest status among all creations. Thanks God to give me strength, patience and ability to complete this study.

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On this occasion of a major work in my life, I like to pay due respect the memory of my beloved father and mother to whom I am indebted for everything of my life. I pray for their salvation.

I am obliged to express my heartiest thanks to my two sisters for their cooperation, nice and understanding behavior during my research work..

Finally, I like to share my satisfaction of completing this task with my well wishers, friends but the responsibility of errors and deficiencies that still remain devolves on me along.

Abstract

Transform methods has their great importance in the field of applied sciences, especially in engineering sciences. To most of us Laplace transform is well known and we are acquainted to solve differential equations with this important tool. But it deals with the continuous variable/analog signals. In this *computer world* we need the tools to deal with discrete variable/digital signals. Unfortunately we have a little knowledge about them i.e. we are not familiar with discrete transforms. The main objective of this thesis was to be familiarized with some discrete transforms. For the purpose Z-transform, which is the most conversant one of the family of discrete transforms is taken. Also discrete counterpart of the Fourier transform, DFT and its calculation technique Fast Fourier Transform (FFT) is considered. Some detail of those transforms has been addressed. Fortunately we have devised a lemma for Z-transform, along with its proof has been presented. Finally a brief introduction to the newest transform, the Wavelet transform is introduced.

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INTRODUCTION

Transform means shift from one form to other. The methods which transforms something from one form to some other form are termed as transform methods. Generally it is required or used to shift variables from one type to other type (e.g $t \rightarrow s$). As variables/parameters have two different forms (e.g. continuous and discrete) so the transform methods will have also two types; one will handle continuous and the other will handle discrete variables/parameters. When the whole of the space is to be considered then for continuous variable one require integration and for discrete variable summation is used. Thus to transform continuous variable integrals are used. We generally use the terminology "Integral Transform" for the purpose. Similarly for discrete variables "Discrete Transform" is used. For both the transforms some Kernel is to be used. For integral transform integration is to be performed over the domain after multiplication by the kernel. In a similar fashion summation is taken over the domain after multiplication by the kernel. On the basis of this kernels the transformed are labeled. Sometimes the domain may be finite, in these cases they are labeled as finite transforms (e.g. Finite Fourier transforms). Transform methods have their own merits in the field of applied sciences, especially in the field of engineering sciences. When a physical system is modeled sometimes differential equations (Ordinary or Partial) arises. For example when a simple circuit is modeled a differential equation is raised. In which inductance, capacitance, resistance and e.m.f. will be present. These differential equation can be solved by general mathematical tools for solving differential equations, but also can be easily solved by Laplace transform method. Because after introducing the Laplace transform to the differential equation one will require some algebraic manipulation and finally the inverse transform will provide the required result. If the initial or boundary conditions were given the arbitrariness present in the solution can be removed to get particular solutions. The Laplace transform is very much useful in solving ordinary differential equation with less effort. If partial differential equations are there (of two independent variables) Laplace transform reduces the form to ordinary differential equations. Which are less tedious than partial differential equations. From these discussion it is clear that Laplace transform is useful tool especially to applied scientist and engineers. In a similar fashion it is observed that when Z-transform is applied to difference equation one get a form which after algebraic manipulation and inverse transform provide the solution of the difference equation. Difference equation arises in case of discrete functions as differential equations arises in case of continuous function. Thus it is observe that transform methods, both integral and discrete, is an essential tool to be familiarized to applied scientist and engineers. In the field of signal processing time and frequency are the matter of interest. So in the field of signal processing both integral transforms and discrete transforms are used. Many common integral transforms used in the field of signal processing have their discrete counterpart (e.g. Fourier and wavelet transforms have their discrete counterparts as Discrete Fourier Transform (DFT), Discrete Sine transform (DST), Discrete Cosine Transform (DCT),Discrete Wavelet Transform (DWT), etc.). There are some other discrete transforms, e.g. Z-transform, Discrete Chebyshev transform, Hadamard transform, Fast Fourier Transform (FFT, a popular implementation of the (DFT), Fast wavelet transform.

With the advent of fast and cheap digital computers, there has been renewed emphasis on the analysis and design of digital systems, which represent a major class of engineering systems. However, it is a mistake to believe that the mathematical basis of this area of work is of such recent vintage. The first comprehensive text in English dealing with difference equations was the treatise of the calculus of Finite Differences due to George Boole and published in 1860. Much of early impetus for the finite calculus was due to the need to carry out interpolation and to approximate derivatives and integrals. Later, numerical methods for the solution of difference equations were devised, many of which were based on finite difference methods, involving the approximation of the derivative terms to produce a difference equation.

Digital systems operate on digital signals, which are usually generated by sampling a continuous-time signal, which is a signal defined for every instant of a possibly infinite time interval. The sampling process generates a discrete-time signal, defined only at the instants when sampling takes place so that a digital sequence is generated. After processing by a computer, the output digital signal may be used to construct a new continuous-time signal, perhaps by the use of a zero-order hold device, and this in turn might be used to control a plant and process.

In many engineering applications the function (signal) under consideration is a continuous function of time that needs to be processed by a digital computer. To do this the continuous time-domain signal $x(t)$ must be sampled at discrete intervals of time. The sample signal $\tilde{x}(t)$ is then processed as an approximation to the true signal $x(t)$.

Let $x(t)$ be an energy-limited continuous-time (analog) signal. If we measure the signal amplitude and record the result at a regular interval *h*, we have a discrete-time signal

$$x(n) = x(t_n), \qquad n = 0,1,2,......, N-1 \text{ where } \qquad t_n = nh$$

For simplicity in writing and convenience of computation, *x*(*n*) is generally used with the sampling period *h* understood. This discretized sample values constitute a signal, called a digital signal.

In order to have a good approximation to a continuous bandlimited function *x*(*t*) from its samples {*x*(*n*)}, the sampling interval *h* must be chosen such that $h \le ® / \Omega$ where 2Ω is the bandwidth of the function *x*(*t*) [i.e., $\hat{x}(\omega) = 0$ means (Fourier transform of $x(t)$ is zero] for all $|\mu| > \Omega$. The choice of *h* above is the Nyquiest sampling rate, and the Shannon recovery formula

$$x(t) = \sum_{n \in \mathbf{Z}} x(nh) \frac{\sin ® (t - nh)}{® (t - nh)}$$

enables us to recover the function *x*(*t*).

The relation between a continuous function *x*(*t*) and its sample values *x*(*kT*), *k*=0, ±1, ±2,….., where *T* is a fixed interval of time, is one of prime importance in digital processing techniques. If the Fourier transform of *x*(*t*) can always be recovered from the knowledge of its sample values *x*(*kT*), provided that the sampling rate is “fast enough” i.e. at a rate that is at least twice the highest significant frequency of the signal. This remarkable result is known as the sampling theorem and plays a central role in digital processing techniques. Functions whose transform is zero everywhere except for a finite interval are known as band-limited waveforms in signal analysis. Such signals do not actually exist in the real world, but theoretical considerations of band limited waveforms are fundamental to the digital field.

In an ideal situation we can assume that sampling is performed instantaneously and thus represent the sampled waveform by

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t)¢(t - kT) = \sum x(kT)¢(t - kT) \qquad (0.1)$$

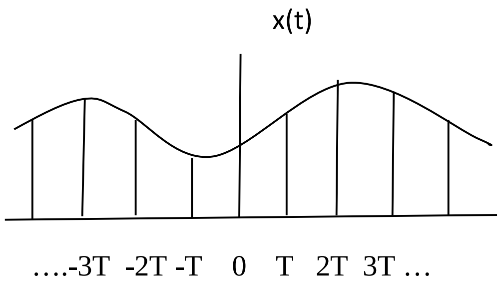

Fig.: Sample function

where $¢(t-kT)$ is the impulse functions. The sampled function is really a train of impulse functions in this sense, but it is otherwise treated as if it were a continuous function of *t*. We recognize (1) as a comb function where the impulses are weighted by the sample values *x*(*kT*). In reality, we cannot obtain an infinite number of samples as suggested in (1).That is, we must always settle for *N* samples over a total time duration *NT*, and in this case, Eq.(1) is approximated by

$$\tilde{x}(t) = \sum_{k=0}^{N-1} x(kT)¢(t-kT) \qquad (0.2)$$

A desired portion of a signal can be removed from the main signal by multiplying the original signal by another function, which is zero outside the interval desired. Let $¤(t)$ be a real-valued window function. Then the product $f_b(t) = f(t)¤(t-b)$ will contain the information of $f(t)$ near $t = b$. The matter will be discussed latter.

Not only analog (continuous) signals are discretized to analyze, but also in the numerical solution of ordinary differential equations, the derivatives are discretized by replacing them by the finite (forward) differences. This gives rise to difference equations of the higher order. Thus a continuous process described by a differential equation is approximated by a discrete process described by its counterpart a difference equation. For example, in a third order ordinary differential equation

$$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = F(x)$$

The derivatives can be replaced by

$$y' = \frac{y_{n+1} - y_n}{h}, \; y'' = \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2}, \; y''' = \frac{y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n}{h^3}$$

which result in a third order differences equation of the form

$$b_3 y_{n+3} + b_2 y_{n+2} + b_1 y_{n+1} + b_0 y_n = F(x)$$

A sequence is a numerical valued function whose domain of definition is the set of integers. It is denoted by $\{a_n\}$ or a_n or $u(n)$. A *k*th order linear difference equation in the sequence y_n is of the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_1 y_{n+1} + a_0 y_n = f(n) \qquad (0.3)$$

where *n*=0,1,2,…Thus (0.3) represents not just a single equation but an infinite system of equations one equation for every *n*. Here the coefficients $a_0, a_1, a_2, \ldots\ldots, a_j \ldots.$ are all constant and do not depend on *n*. Here $f(n)$ depends only on *n*. When a_k is chosen as one (0.3) is said to be in the standard form. If $f_n \neq 0$ for all $n,$ then (0.3) is said to be non-homogeneous, otherwise it is said to be homogeneous. The order of the difference equation (0.3) is the positive integer *k* which is the greatest difference in the index of non-zero values of y. Equation (0.3) is linear because each term in one (0.3) is the first degree (linear) in y_n. Thus (0.3) is non-homogeneous *k*th order linear difference equation with constant coefficients.

Difference equation is also referred to as recurrence relation since it is also referred to as recurrence relation since it expresses y_{n+k} in terms of one or more of the previous terms (of the sequence) namely $y_{n+k-1}, \ldots, y_{n+1}, y_n$. In this case (0.3) can be written as $y_{n+k} = -a_{k-1} y_{n+k-1} \cdots - a_1 y_{n+1} - a_0 y_n + f(n)$. The difference equation (0.3) models a physical system. So f_n is known as system input (system excitation or forcing sequence or driving sequence) while y_n is referred to as system output (system response). The structure of the system is defined by the values of the coefficients and order of the equation. Thus any system output depends on the system input and the structure of the system. The general solution of (0.3) determines the output y_n which depends only on n (but no longer on the prior terms of the sequence) and describes the complete sequence y_n in the closed form. Thus any sequence y_n that satisfies the difference equation (0.3) is a solution of (0.3).

First order homogeneous difference equation

To proceed to solve a first order linear difference equation $y_{n+1} \quad by_n = 0$ for $n \geq 0$ and b is a constant with boundary condition $y_0 = d$, let the solution be $y_n = r^n$ with $r \neq 0$. Then $y_{n+1} = r^{n+1}$. Substituting these in the given difference equation, we have $r^{n+1} - br^n = 0 \Rightarrow r = b$

Thus the general solution of the difference equation $y_{n+1} - by_n = 0$ is given by $y_n = cb^n$ (since if b^n is a solution then any non-zero constant multiple of it is also a solution). In addition as boundary condition is $y_0 = d$ then $d = y_0 = cb^0 \therefore c = d$. Then the particular solution is $y_n = db^n$.The solution y_n defines a discrete function whose domain is the set □ of all non-negative integers.

Second order linear homogeneous difference equation with constant coefficients

Let us consider $a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0$ (0.4)

Let us assume (as earlier) $y_n = r^n$, $r \neq 0$ (0.5)

as a solution of (0.4). Then substituting (0.5) in (0.4), we get $a_2 r^{n+2} + a_1 r^{n+1} + a_0 r^n = 0$

$\Rightarrow a_2 r^2 + a_1 r + a_0 = 0$

Thus (0.5) is solution of (0.4) if $a_2 r^2 + a_1 r + a_0 = 0$ (0.6)

The equation (0.6) which is a quadratic in *r* is known as the characteristic/auxiliary equation of (0.4). Let the roots of this equation be r_1 and r_2. Three cases may arise.

Case 1: <u>When the roots are real and district</u>

In this case clearly $r_1^n$and $r_2^n$are two linearly independent solutions. Thus the general of (0.4) will be the linear combinations of them, i.e.

$$y_n = c_1 r_1^n + c_2 r_2^n$$

Case 2: When the roots are real and equal (say r)

In these case r^n and nr^n will be two different solutions. Hence the general solution in this case will be $y_n = (c_1 + c_2 n) r^n$

Case 3: <u>When the roots are complex</u>

Since the complex roots occurs in pair, let the roots are given by $a \pm ib$. Then the general solution will take the form $y_n = r^n \left(c_1 \cos n^{-} + c_2 \sin n^{-}\right)$, $r = \sqrt{a^2 + b^2}$ and $^{-} = \tan^{-1}\frac{b}{a}$

This analysis can be extended to *k*th order difference equation by considering the nature of the *k* roots of the auxiliary equation which will be a *k*th degree polynomial.

Before proceeding to non-homogeneous difference equations let us recollect the followings:

- The forward-difference or advancing difference operator Δ is defined by $\Delta f_k = f_{k+1} - f_k$
- The shift operator E is defined as the operator that increases the argument of a function by one tabular interval. Thus $Ef_k = Ef(x_k) = f(x_k + h) = f(x_{k+1}) = f_{k+1}$
- Δ and E are related $E = 1 + \Delta$.

The difference equation

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_1 y_{n+1} + a_0 y_n = f(n) \qquad (0.3)$$

can be written in terms of E as follows

$$(a_k E^k + a_{k-1} E^{k-1} + \cdots + a_1 E + a_0) y_n = f(n) \qquad (0.7)$$

Non-homogeneous Equations

The general solution of a non-homogeneous linear difference equation with constant coefficients (0.3) is the sum of the complementary function and any particular solution. Here the complementary function (C.F.) of (0.3) is the general solution of the corresponding homogeneous equation (0.4). Particular solution, more often known as particular integral (P.I.) of (0.3), can be obtained by (a) method of undetermined coefficients (b) short cut inverse operator methods.

(a) Method of undetermined coefficients

The particular integral is assumed in a particular form depending on the form of the RHS function f_n. On the basis of the RHS functions are chosen and after taking their linear combination P.I. is formed. That P.I. is substituted on the LHS and comparing the coefficients are calculated.

(b) Inverse operator methods

The non-homogeneous equation (8) can be written as

$$F(E)y_n = f(n) \tag{0.8}$$

where $F(E) = (a_k E^k + a_{k-1}E^{k-1} + \cdots + a_1 E + a_0)$ is a function of the operator *E*. Then P.I.$= \frac{1}{F(E)} f(n)$

Case 1: If $f(n) = a^n$ then

P.I.$= \frac{1}{F(E)} a^n = \frac{1}{F(a)} a^n$, provided $F(a) \neq 0$.

$$\text{P.I.} = \frac{1}{(E-a)^k} a^n = \frac{n(n-1)(n-2)....(n-\overline{k-1})}{\text{k!}} a^{n-k}$$

Case 2: If $f(n) = \sin \alpha n$ then

$$\text{P.I.} = \frac{1}{F(E)} \sin \sim n = \frac{1}{F(E)} \left(\frac{e^{in\sim} - e^{-in\sim}}{2i} \right) = \frac{1}{2i} \left(\frac{1}{F(E)} a^n - \frac{1}{F(E)} b^n \right)$$

where $a = e^{i\sim}$ and $b = e^{-i\sim}$.

Similarly if $f(n) = \cos \alpha n$, then

$$\text{P.I.} = \frac{1}{F(E)} \cos \sim n = \frac{1}{F(E)} \left(\frac{e^{in\sim} + e^{-in\sim}}{2} \right) = \frac{1}{2} \left(\frac{1}{F(E)} a^n + \frac{1}{F(E)} b^n \right)$$

Case 4: If $f(n) = n^m$ or polynomial in n. Replace E by $1 + \Delta$ and expand $1/F(1 + \Delta)$ in binomial series in ascending powers of Δ up to Δ^m .Express $f(n)$ in a factorials and use $\Delta[x]^n = n[x]^{n-1}$

Case 5: If $f(n) = a^n V(n)$where $V(n)$ is polynomial in n. Then

P.I$= \frac{1}{F(E)}\{a^n V(n)\} = a^n \frac{1}{F(aE)} V(n)$

It is clear that the discrete transforms have their great importance in the field of signal processing but a little is known to us about them. Especially at the undergraduate level a very little information is provided to the students about them. Also some new transforms are emerging which may have their uses in the field of signal processing, which also include signal compression, pattern recognition etc. Though the main objective of this research is to make familiarize the different discrete transforms, we have devised a corollary in the properties of z-transform. The scope of utilizations of the existing discrete transforms will also be sorted.

This thesis will address Z-transform (Chapter-2), Discrete Fourier transforms (Chapter-3) and a brief introduction to Wavelet transform (Chapter-4).

CHAPTER-2

Z-TRANSFORM

Before the discussion of the main topic some related topics will be addressed.

Discrete-time signal and systems

A discrete signal has values which are defined only at discrete values of time or some other appropriate variable, for example space. Such a signal may be generated by sampling a continuous-time signal at regular time intervals n , n=0,1,…,where T is sampling period. Thus if the analog input signal $x(t)=e^{-at}$ is applied to a digital filter, it will give rise the sequence $x(n)=\sum_{n=-\infty}^{\infty} x(t)¢\left(t-nT\right)$. For $t=nT$, the sampled signal sequence is $x(n)=\left[e^{0},e^{-aT},e^{-2aT},.......\right]$.

Discrete signal may also be generated, artificially via some algorithm in a computer. The amplitude of a discrete-time signal may have discrete values (discrete time, discrete amplitude), or it may be continuous.

By tradition, a discrete-time signal is represented as a sequence of numbers:

$x(n)$, n=0,1,… (2.1a)

$x(n\ )$, n=0,1,… (2.1b)

x_n, n=0,1,… (2.1c)

Where the symbol $x(n)$, $x(n\ )$ or x_n indicates the value of the signal at the discrete time n (or n).For convenience we will use the symbol $x(n)$ to denote both the value of the sequence at the discrete time n and the sequence itself unless we wish to emphasize the difference. The meaning will be clear from the context.

Let $x(t)$ be an energy-limited continuous-time (analog) signal. If we measure the signal amplitude and record the result at a regular interval *h*, we have a discrete-time signal

$x(n) = x(t_n), \qquad n = 0,1,2,......, N-1$ where $\qquad t_n = nh$

For simplicity in writing and convenience of computation, $x(n)$ is generally used with the sampling period h understood. This discretized sample values constitute a signal, called a digital signal.

In order to have a good approximation to a continuous band limited function $x(t)$ from its samples $\{x(n)\}$, the sampling interval h must be chosen such that $h \le ® / \Omega$ where 2Ω is the bandwidth of the function $x(t)$ [i.e.,$\hat{x}(\omega) = 0$ means (Fourier transform of $x(t)$ is zero] for all $|\mu| > \Omega$. The choice of h above is the Nyquiest sampling rate, and the Shannon recovery formula

$$x(t) = \sum_{n \in \mathbf{Z}} x(nh) \frac{\sin ® (t - nh)}{® (t - nh)}$$

enables us to recover the function $x(t)$.

A discrete-time is essentially mathematical algorithm that takes an input sequence, $x(n)$, and produces an output sequence, $y(n)$. Example of discrete-time systems are digital controllers, digital spectrum analyzers, and digital filters. A discrete-time system may be linear or nonlinear, time invariant or time varying. Linear time-invariant (LTI) systems form an important class of systems used in DSP.

A discrete-time system is said to be linear if it obeys the principles of superposition. That is, the response of a linear to two or more inputs is equal to the sum of the response of the systems to each input acting separately in the absence of all the other inputs is equal to the sum of the response of the system to each input acting separately in the absence of all the other inputs. For example, if an input $x_1(n)$ to the system gives rise to the output $y_1(n)$, and another input $x_2(n)$, produces the output $y_2(n)$, the response of the system to both inputs will be

$$a_1 x_1(n) + a_2 x_2(n) \rightarrow a_1 y_1(n) + a_2 y_2(n) \qquad (2.2)$$

where $a_1$and a_2 are arbitrary constants.

A discrete-time system is said to be time invariant (sometimes referred to as shift invariant) if its output is independent of the time the input is applied. For example, if the input $x(n)$ gives the output $y(n)$,then the input $x(n-k)$ will give the output $y(n-k)$:

$$x(n) \to y(n) \tag{2.3a}$$

$$x(n-k) \to y(n-k) \tag{2.3b}$$

That is, a delay in the input causes a delay by the same amount in the output signal. The input-output relationship of an LTI system is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \tag{2.4}$$

where $h(k)$ is the impulse response of the system. The values of $h(k)$ completely define the discrete-time system in the time domain. An LTI system is stable if its impulse response satisfies the condition

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{2.5}$$

This condition is satisfied if $h(k)$ is of finite duration or if $h(k)$ decays towards zero as k increases.

A causal system is one which produces an output only when there is an input. All physical systems are casual. In general, a casual discrete-time sequence, $x(n)$, or the impulse response, $h(k)$, of a discrete-time system is zero before time 0,that is $x(n)$ =0, $n < 0, k < 0$

The Laplace transform plays a very important role in the analysis of analog signals or systems and in solving linear constant coefficient differential equations. It transforms the differential equations into the complex s-plane where algebraic operations and inverse transform can be performed to obtain the solution.

Like the Laplace transform, the z-transform provides the solution for linear constant coefficient difference equations, relating the input and output digital signals in the time domain. It gives a method for the analysis of discrete time systems in the frequency domain.

The analysis of any sampled signal or sampled data system in the frequency domain is extremely difficult using s-plane representation because the signal or system equations will contain infinite long polynomials due to the characteristic infinite number of poles and zeros. Fortunately this problem may be overcome by using the z-transform, which reduces the poles and zeros to a finite number in the z-plane.

The purpose of the z-transform is to map (transform) any point $s = \pm\pm \pm i\mu$ in the s-plane to a corresponding point z($r \llcorner \theta$) in the z-plane by the relationship $z = e^{sT}$ where T is sampling period (seconds)

Under this mapping, the imaginary axis, $\pm = 0$ maps on the unit circle $|z| = 1$ in the z-plane. Also, the left hand half-plane $\sigma < 0$ corresponds to the interior of the unit circle $|z| = 1$ in the z-plane. Considering that the real part of x is zero, i.e. $\pm = 0$ we have $z = e^{i\mu T} = 1\angle \pm i\mu T$ which gives the values of z (in polar form) shown as in the following table.

$\pm = 0,\ \mu_s = \frac{2®}{T}$									
$i\mu$	0	$\mu_s/8$	$\mu_s/4$	$3\mu_s/8$	$\mu_s/2$	$5\mu_s/8$	$3\mu_s/4$	$7\mu_s/8$	μ_s
$z = 1\angle\mu T$	$1\angle 0^\circ$	$1\angle 45^\circ$	$1\angle 90^\circ$	$1\angle 135^\circ$	$1\angle 180^\circ$	$1\angle 225^\circ$	$1\angle 270^\circ$	$1\angle 315^\circ$	$1\angle 360^\circ$

The z-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems. For example, we shall see that in the z-domain (complex z-plane) the convolution of two time-domain signals is equivalent to multiplication of their corresponding z-transforms. This property greatly simplifies the analysis of the response of an LTI system to various signals. In addition, the z-transform provides us with a means of characteristic an LTI system, and its response to various signals, by its pole-zero locations.

The transform is used to characterize signals in terms of their pole-zero patterns. The z-transform of a signal is used to obtain the time-domain representation of the signal. The one-side z-transform is used to solve linear difference equations with nonzero initial conditions.

2.1 The Direct z-transform

The z-transform of a discrete-time signal x(n) is defined as the power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (2.6)$$

where z is a complex variable. The relation (2.6) is sometimes called the direct z-transform because it transforms the time-domain signal x(n) into its complex plane representation X(z). The inverse procedure [i.e., obtaining x(n) from X(z)] is called the inverse z-transform .

For a convenience, the z-transform of a signal x(n) is denoted by

$$X(z) \equiv Z\{x(n)\} \tag{2.7}$$

Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The region of converges. The region of convergence (ROC) of X(z) attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC.

Let us express the complex variable z in polar form as

$$z = re^{i\theta} \tag{2.8}$$

where $r = |z|$ and $\theta = \measuredangle z$. Then X(z) can be expressed as

$$X(z)\Big|_{z=re^{i\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-in\theta}$$

In the ROC of X(z), $|X(z)| < \infty$. But

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-in\theta} \right| \le \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} e^{-in\theta} \right| = \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right|$$

Hence $|X(z)|$ is finite if the sequence $x(n)r^{-n}$ is absolutely summable.

The problem of finding the ROC for X(z) is equivalent to determining the range of values of r for which the sequence $x(n)r^{-n}$ is absolutely summable. To elaborate, let us rewrite the above equation as

$$|X(z)| \le \sum_{n=-\infty}^{-1} \left| x(n) r^{-n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \le \sum_{n=1}^{\infty} \left| x(-n) r^{n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

If X(z) converges in some region of the complex plane, both summations of the above equation must be finite in that region. If the first sum of the above converges, there must exist values of r small enough such that the product sequence $x(-n)r^{n}$, $1 \le n \le \infty$,is absolutely summable. Therefore, the ROC for the first sum consists of all points in a circle of some radius $r_1$where $r_1 < \infty$. On the other hand, if the second sum converges, there must exist values of r large enough such that the product sequence $x(n)/r^n$, $0 \le n \le \infty$ is absolutely summable. Hence the ROC for the second sum consists of all points outside a circle of radius $> r_2$.

Since the convergence of X(z) requires that both sums be finite, it follows that the ROC of X(z) is generally specified as the annular region in the z-plane, $r_2 < r < r_1$., which is the common region where both sums are finite. On the other hand, if $r_2 > r_1$, there is no common region of convergence for the to sums and hence X(z) does not exist.

2.2 Importance Properties of the ROC for the z-transform

(i) The ROC does not contain any poles.

(ii) When *x*(*n*) is of finite duration, then the ROC is the entire z-plane, except possibly z=0 and/or z=∞.

(iii) If *x*(*n*) is a right-sided sequence, the ROC will not include infinity.

(iv) If *x*(*n*) is a left-sided sequence, the ROC will not include z=0.However, if *x*(*n*)=0 for all *n*>0, the ROC will include z=0.

(v) If *x*(*n*) is two-sided, and if the circle |z|=r_0 is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle |z|=r_0. That is the ROC includes the intersection of the ROC's of the components.

(vi) If X(z) is rational, then the ROC extends to infinity, i.e. the ROC is bounded by poles.

(vii) If *x*(*n*) is causal, then the ROC includes z=∞.

(viii) If *x*(*n*) is anti-causal, then the ROC includes z=0.

2.3 THE ONE-SIDED Z-TRNSFORM

The two sided z-transform requires that the corresponding signals be specified for the entire time range$-\infty < n < \infty$. This requirement prevents its use for a very useful family of practical problems, namely the evaluation of the output of non-relaxed systems. As we recall, this systems are described by difference equations with nonzero initial conditions. Since the input is applied at a finite time, say n_0, both input and output signals are specified for $n \ge n_0$, but by no means are zero for $n < n_0$. Thus the two-sided z-transform cannot be used.

2.3.1 Definition and properties

The one-sided or unilateral z-transform of a signal *x*(*n*) is defined by

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n} \quad \text{i.e. } Z\{x(n)\}=X(z)$$

No confusion will arise as in this case *n* will take only non-negative integral values, whereas in the case of direct z-transform *n* can take both negative and positive integral values. The one-sided z-transform differs from the two-sided transform in the lower limit of the summation, which is always zero, whether or not the signal *x*(*n*) is zero for $n<0$ (i.e., causal). Due to this choice of lower limit, the one-sided z-transform has the following characteristics:

- It does not contain information about the signal *x*(*n*)for negative values of time (i.e., for an n<0).
- It is unique only for causal signals, because only these signals are zero for n<0.
- The one-sided z-transform $X(z)$ of *x*(*n*) is identical to the two-sided z-transform of the signal $x(n)u(n)$. Since $x(n)u(n)$ is causal, the ROC of its transform, and hence the ROC of $X(z)$ is always the exterior of a circle. Thus when we deal with one-sided z-transforms, it is not necessary to refer to their ROC.

2.3 The Inverse z-transform

Often, we have the z-transform $X(z)$ of a signal and we must determine the signal sequence. The inverse z-transform (IZT) allows us to recover the discrete-time sequence $x(n)$, given its z-transform. The procedure for transforming from the z-domain to the time domain is called the inverse z-transform. Symbolically, the inverse z-transform may be defined as

$$x(n)=Z^{-1}[X(z)] \qquad (2.9)$$

where $X(z)$ is the z-transform of $x(n)$ and $Z^{-1}$is the symbol for the inverse z-transform.

The mathematical basis for obtaining $x(n)$ from $X(z)$ can be derive by using the Cauchy integral theorem, as *z* is a complex variable.

Since $X(z)=\sum_{k=-\infty}^{\infty}x(n)z^{-n}$, let us multiply both sides of by z^{n-1} and integrate both sides over a closed contour within the ROC of $X(z)$.which encloses the origin. Thus we have……..

$$\oint_c X(z)z^{n-1}dz=\oint_c \sum_{k=-\infty}^{\infty}x(k)z^{n-1-k}dz \qquad (2.10)$$

where C denotes the closed contour in the ROC of $X(z)$, taken in a counter clock-wise direction. Since the series converges on this contour, we can interchange the order of integration and summation on the right-hand side of (2.10). Thus (2.10) becomes

$$\oint_c X(z) z^{n-1} dz = \sum_{k=-\infty}^{\infty} \oint_c x(k) z^{n-1-k} dz \qquad (2.11)$$

Using the Cauchy integral theorem, which states that

$$\frac{1}{2\pi i} \oint_c z^{n-1-k} dz = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases} \qquad (2.12)$$

where C is any contour that encloses the origin. By applying (2.12), the right-hand side of (2.11) reduces to $2\pi i\, x(n)$ and hence the desired inversion formula be

$$x(n) = \frac{1}{2\pi i} \oint_c X(z) z^{n-1} dz \qquad (2.13)$$

Although the contour integral in (2.13) provides the desired inversion formula for determining the sequence x(n) from the z-transform, it is not generally used to obtain inverse z-transforms.

In practice ,$X(z)$ is often expressed as a ratio of two polynomials in z^{-1} or equivalently in z:

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}} \qquad (2.14)$$

In this form, the inverse z-transform, $x(n)$, may be obtained using one of several methods including the following three:

(1) Power series expansion method;

(2) Partial fraction expansion method,

(3) Residue method.

Each method has its own merits and demerits. In terms of mathematical rigour, the residue method is perhaps the most elegant. The power series method, however, lends itself most easily to computer implementation.

2.3.1 Power series method

Given the z-transform, $X(z)$, of a casual sequence as in Equation (2.14), it can be expanded into an infinite series in z^{-1} or z by long division (also called synthetic division):

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots \quad (2.15)$$

In this method, the numerator and denominator of $X(z)$ are first expressed in either descending powers of z ascending powers of z^{-1} and the quotient is then obtained by long division.

The long division approach provides us the following relations:

$x(0) = b_0 / a_0$; $x(1) = \left[b_1 - x(0)a_1\right] / a_0$; $x(2) = \left[b_2 - x(1)a_1 - x(0)a_2\right] / a_0$; $x(3) = \left[b_3 - x(2)a_1 - x(1)a_2 - x(0)a_3\right] / a_0$ ………… $x(n) = \left[b_n - \sum_{i=1}^{n} x(n-i)a_i\right] / a_0$.

Thus we have $x(n) = \left[b_n - \sum_{i=1}^{n} x(n-i)a_i\right] / a_0$ for $n \geq 1$ and $x(0) = b_0 / a_0$

2.3.2 Partial fraction expansion method

In this method, the z-transform is first expanded into a sum of simple partial fractions. The inverse z-transform of partial fraction is then obtained from tables (such a table is presented as Table 2.1) and then summed to give the overall inverse z-transform. As has been considered earlier, let we have given

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}} \quad (2.16)$$

If the poles of $X(z)$ are of first order and $N = M$, then $X(z)$ can be expanded as

$$X(z) = B_0 + \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_2}{1 - p_2 z^{-1}} + \cdots + \frac{C_M}{1 - p_M z^{-1}}$$

$$= B_0 + \frac{C_1 z}{z - p_1} + \frac{C_2 z}{z - p_2} + \ldots + \frac{C_M z}{z - p_M} = B_0 + \sum_{k=1}^{M} \frac{C_k z}{z - p_k} \quad (2.17)$$

where p_k are the poles of $X(z)$ (assumed distinct), C_k are the partial fraction coefficients and

$$B_0 = b_N / a_N \quad (2.18)$$

The C_k are also known as the residues of $X(z)$, by definition.

If the order of the numerator is less than that of the denominator in Equation (2.16), that is $N < M$, then B_0 will be zero. If $N > M$ then $X(z)$ must be reduced first, to make $N \leq M$, by long division with the numerator and denominator polynomials written in descending powers of z^{-1}. The reminder can be then be expressed as in Equation (2.17).

The coefficient, C_k, associated with the pole p_k may be obtained by multiplying both sides of Equation 4.15 by $(z - p_k)/z$ and $z = p_k$:

$$C_k = \frac{X(z)}{z}(z - p_k)\Big|_{z=p_k}$$

If X(z) contains one or more multiple-order poles (that is poles that are coincident) then extra terms are required in equation (2.17) to take this into account. For example, if X(z) contains an mth-order pole at $z = p_k$ the partial fraction expansion must include terms of the form

$$\sum_{i=1}^{m} \frac{D_i}{(z - p_k)^i}$$

The coefficients, D_i, may be obtained from the relationship

$$D_i = \frac{1}{(m-i)!} \frac{d^{m-i}}{dz^{m-i}} \left[(z - p_k)^m \frac{X(z)}{z} \right]_{z=p_k}$$

2.3.3 Residue method

In this method the IZT is obtained by evaluating the contour integral

$$x(n) = \frac{1}{2®i} \oint_C z^{n-1} X(z)\, dz \qquad (2.19)$$

where *C* is the path of integration enclosing all poles of *X(z)*. For rational polynomials, the contour integral in equation (2.19) is evaluated using a fundamental result in complex variable theory known as Cauchy's residue theorem :

$$x(n) = \frac{1}{2®i} \oint_C z^{n-1} X(z)\, dz$$

= sum of the residues of $z^{n-1}X(z)$ at all the poles inside C.

In the last section, it was stated that the partial fraction coefficients, the C_k , are also referred to as residues of *X(z)* and a way of obtaining their values was given. The key point to remember is that every residue, C_k, is associated with a pole, p_k. In the present method, the residue of $z^{n-1}X(z)$ at the pole p_k is given by

$$\text{Res}\left[F(z), p_k\right] = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-p_k)F(z)\right]_{z=p_k} \tag{2.20}$$

where $F(z) = z^{n-1}X(z)$, m is the order of the pole at p_k and $\text{Res}\left[F(z), p_k\right]$ is the residue of $F(z)$ at p_k. For a simple (distinct) pole, equation (2.20) reduces to

$$\text{Res}\left[F(z), p_k\right] = (z-p_k)F(z) = (z-p_k)z^{n-1}X(z)\Big|_{z=p_k} \tag{2.21}$$

2.4 PROPERTIES OF Z-TRANSFORM

(i) Linearity

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$ then

$Z\{a_1x_1(n) \pm a_2x_2(n)\} = a_1X_1(z) \pm a_2X_2(z)$

(ii) Time Shifting

a) If $Z\{x(n)\} = X(z)$ then $Z\{x(n-k)\} = z^{-k}X(z)$

b) If $Z\{x(n)\} = X(z)$ then

$Z\{x(n+k)\} = z^k\left[X(z) - x_0 - x_1z^{-1} - x_2z^{-2} - .....x_{k-1}z^{k-1}\right]$

(iii) Scaling in z-domain

If $Z\{x(n)\} = X(z)$ with $ROC: r_1 < |z| < r_2$ then $Z\{a^nx(n)\} = X(a^{-1}z)$ with $ROC: |a|r_1 < |z| < |a|r_2$

(iv) Time reversal

If $Z\{x(n)\} = X(z)$ with $ROC: r_1 < |z| < r_2$ then $Z\{x(-n)\} = X(z^{-1})$ with $ROC: \frac{1}{r_1} < |z| < \frac{1}{r_2}$

(v) Differentiation in the z-domain or multiplication effect of *n*

If $Z\{x(n)\} = X(z)$ then $Z\{nx(n)\} = -z\frac{dX(z)}{dz}$

(vi) Convolution of two sequences

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$ then

$$Z\left\{x_1(n) * x_2(n)\right\} = X(z) = X_1(z)\,X_2(z)$$

(vii) Correlation of two sequences

If $Z\left\{x_1(n)\right\} = X_1(z)$ and $Z\left\{x_2(n)\right\} = X_2(z)$ then

$$Z\left\{\sum_{n=-\infty}^{\infty} x_1(n)\,x_2(n-l)\right\} = X_1(z)\,X_2\left(z^{-1}\right)$$

(viii) Multiplication of two sequence

If $Z\left\{x_1(n)\right\} = X_1(z)$ and $Z\left\{x_2(n)\right\} = X_2(z)$ then

$$Z\left\{x_1(n)\,x_2(n)\right\} = X(z) = \frac{1}{2 \circledR i}\int X_1(v)\,X_2\left(\frac{z}{v}\right)v^{-1}dv$$

(ix) Parseval's relation

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$Z\left\{\sum_{n=-\infty}^{\infty} x_1(n)\,x_2^*(n)\right\} = \frac{1}{2 \circledR i}\int X_1(v)\,X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

(x) Initial value theorem

If $Z\left\{x(n)\right\} = X(z)$, $x(n)$ is causal [i.e., $x(n) = 0$ for $n < 0$], then $x(0) = \lim_{z\to\infty} X(z)$

(xi) Final value theorem.

If $Z\left\{x(n)\right\} = X(z)$ then $\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)\,X(z)$

Corollary-I: As $Z\left\{a^n\right\} = \dfrac{1}{1-a/z}$ then

$$Z\left\{\frac{n(n-1)(n-2)...(n-\overline{m-1})}{m!}a^{n-m}\right\} = \frac{d^m}{da^m}\left[\frac{1}{1-a/z}\right];\ n \ge m$$

Proof: From the definition $Z\left\{a^n\right\} = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + az^{-1} + a^2 z^{-2} + ... = \left(1-az^{-1}\right)^{-1} = \dfrac{1}{1-a/z}$.

i.e. $Z\left\{a^n\right\} = \dfrac{1}{1-a/z}$.

If we consider *a* as parameter then we can differentiate both sides with respect to *a*, thus we will have $\frac{d}{da}\left[Z\left\{a^n\right\}\right]=\frac{d}{da}\left[\frac{1}{1-a/z}\right]$. Upon interchanging the differentiation and transformation operator we get $Z\left\{na^{n-1}\right\}=\frac{1}{z(1-a/z)^2};\ n\ge 1$

The above result can be easily verified using the properties, as follows:

$$Z\left\{a^{n-1}\right\}=\sum_{n=1}^{\infty}a^{n-1}z^{-n}=z^{-1}+az^{-2}+a^2z^{-3}+......$$
$$=z^{-1}\left(1+az^{-1}+a^2z^{-2}+....\right)=\frac{1}{z\left(1-a/z\right)}=\frac{1}{z-a}$$

So $Z\left\{na^{n-1}\right\}$ can be obtained using the *multiplication effect of n*. Thus

$$Z\left\{na^{n-1}\right\}=-z\frac{d}{dz}\left[\frac{1}{z-a}\right]=\frac{z}{\left(z-a\right)^2}=\frac{1}{z\left(1-a/z\right)^2};\ n\ge 1.$$

Continuing the process *m* times and after simplification we will get the result.

Table 2.1: Some important Z-transform, with ROC

$x(n)$	$X(z)$	ROC
$¢(n)$	1	Entire z-plain
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
$e^{\sim n}u(n)$	$\frac{1}{1-e^{\sim}z^{-1}}$	$\lvert z\rvert>\lvert e^{\sim}\rvert$
$(\cos\quad n)u(n)$	$\frac{1-z^{-1}\cos}{1-z^{-1}\cos\quad+z^{-2}}$	$\lvert z\rvert>1$
$(\sin\quad n)u(n)$	$\frac{z^{-1}\sin}{1-z^{-1}\cos\quad+z^{-2}}$	$\lvert z\rvert>1$

Now some example will be presented.

Problem 2.1: Find the z-transform and indicate the ROC of the following problems

(i) $x(n) = \{2, 5, 3, 4, 9\}$

(ii) $x(n) = \left\{ 1, -1, \underset{\uparrow}{2}, 5, 7 \right\}$

(iii) $x(n) = \text{¢}(n)$

(iv) $x(n) = \text{¢}(n-k)$

(v) $x(n) = \text{¢}(n+k)$

(vi) $x(n) = u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$

Solution:

(i) Given that,

$$x(n) = \{2, 5, 3, 4, 9\}$$

Here, $x(1) = 2$, $x(2) = 5$, $x(3) = 3$, $x(4) = 4$, $x(5) = 9$,

We know, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4}$$

$$= 2 + 5z^{-1} + 3z^{-2} + 4z^{-3} + 9z^{-4}$$

ROC: Entire z-plane except $z = 0$.

(ii) Given that, $x(n) = \left\{ 1, -1, \underset{\uparrow}{2}, 5, 7 \right\}$

Here, $x(-2) = 1$, $x(-1) = -1$, $x(0) = 2$, $x(1) = 5$, $x(2) = 7$,

We know, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=-2}^{2} x(n) z^{-n}$$

$$= x(-2) z^2 + x(-1) z + x(0) z^0 + x(1) z^{-1} + x(2) z^{-2}$$

$$= z^2 - z + 2 + 5z^{-1} + 7z^{-2}$$

ROC: Entire z-plane except $z = 0$ and $z = \infty$

(iii) Given that, $x(n) = \delta(n) = \{1, 0, 0, 0.....\}$

We know, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{x(n)\} = Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$= 1z^0 + 0z^{-1} + \ldots.$, since $\delta(n) = \{1, 0, 0, 0.....\}$

$= 1$

ROC: Entire z-plane .

(iv) Given that,

$$x(n) = \delta(n-k)$$

We know, if $Z\{x(n)\} = X(z)$ then $Z\{x(n-k)\} = z^{-k} X(z)$ and $Z\{\delta(n)\} = 1$

$$Z\{\delta(n-k)\} = z^{-k}.1 = z^{-k}$$

ROC: Entire z-plane except $z = 0$.

(v) From the definition

$$Z\{\delta(n+k)\} = \sum_{n=-\infty}^{\infty} \delta(n+k) z^{-n} = 0 + 0 + ... + 1.z^{-(-k)} + 0 + 0 + ... = z^k$$

Since $\delta(m) = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$, thus $\delta(n-k) = \begin{cases} 1, & n = -k \\ 0, & n \neq -k \end{cases}$

ROC: Entire z-plane except $z = \infty$

(vi) Given that, $x(n) = u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$

We know, $u(n) = \{1, 1, 1, ...\}$

Thus $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} u(n) z^{-n} = ...0 + 0 + 1.z^0 + 1.z^{-1} + 1.z^{-2} + ...$

$$= 1 + z^{-1} + z^{-2} + ... = (1 - z^{-1})^{-1} = \frac{z}{1-z}$$

ROC: The ROC is the interior part of the circle $|z| = 1$ i.e, where $|z| < 1$

Problem 2.2: Find the z-transform of $x(n) = a^n u(n) + b^n u(-n-1)$

Solution: We know, $u(m) = \begin{cases} 0 & m < 0 \\ 1 & m \geq 0 \end{cases}$ **,** thus $u(-n-1) = \begin{cases} 0 & n > -1 \\ 1 & n \leq 1 \end{cases}$

and $Z\{x(n)\} = \sum_{n=\infty}^{\infty} x(n) z^{-n}$

let, $x(n) = x_1(n) + x_2(n)$

$\therefore \quad Z\{x(n)\} = Z\{x_1(n)\} + Z\{x_2(n)\}$**,** from linearity property

$$= Z\{a^n u(n)\} + Z\{b^n u(-n-1)\}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=1}^{\infty} b^{-n} z^n$$

$$= \left(1 + az^{-1} + a^2 z^{-2} + ...\right) + \left(b^{-1}z + b^{-2}z^2 + b^{-3}z^3 + ...\right)$$

$$= \left(1 + az^{-1} + a^2 z^{-2} + ...\right) + b^{-1}z\left(1 + b^{-1}z + b^{-2}z^2 + ...\right)$$

$$= \frac{1}{1 - az^{-1}} + \frac{b^{-1}z}{1 - b^{-1}z} \quad \text{provided } |az^{-1}| < 1 \text{ and } |b^{-1}z| < 1$$

The first condition requires that $|z| > |a|$ and that for the second is $|z| < |b|$. If both the conditions are not satisfied simultaneously then we will not get the required transform. Both the conditions can only be satisfied if $|a| < |b|$, thus z will lie within an annular region. In this case z-transform will exists and ROC be $|a| < z < |b|$.

Problem 2.3: Find the z-transform of

(i) $x(n) = u(-n)$ (ii) $x(n) = na^n u(n)$

Solution:

(i) From the definition we have

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} u(-n) z^{-n} = \sum_{n=-\infty}^{0} z^{-n} = \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z}; \qquad |z<1|$$

(ii) To determine the z-transform we first try to find the z-transform of $a^n u(n)$. From the definition we have,

$$Z\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + az^{-1} + az^{-2} + ... = \left(1 - az^{-1}\right)^{-1}$$

$$= \frac{1}{1-az^{-1}} \text{ , provided } \left|az^{-1}\right| < 1 \text{ i.e., } |z| > |a|$$

But we know if $Z\{x\}(n) = X(z)$ then $Z\{x(n)\} = -z\frac{d}{dz}X(z)$

$$\therefore Z\{na^n u(n)\} = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad \text{with the condition } |z| > |a|$$

Problem 2.4: Find the inverse z-transform of $\log\left(1+az^{-1}\right), \quad |z| > |a|$

Solution:

Let, $\quad X(z) = \log\left(1+az^{-1}\right)$

$$\therefore \frac{d}{dz}X(z) = -\frac{az^{-2}}{1+az^{-1}}$$

or, $-z\frac{d}{dz}X(z) = -\frac{az^{-1}}{1+az^{-1}}$

Again $Z\{a^n u(n)\} = \frac{1}{1-az^{-1}}$

$$\therefore \quad Z\left\{(-a)^n u(n)\right\} = \frac{1}{1+az^{-1}}$$

$$\Rightarrow Z\left\{(-a)^{n-1}u(n-1)\right\} = z^{-1}\frac{1}{1+az^{-1}} \qquad n \geq 1$$

$$\therefore \quad Z\left\{a(-a)^{n-1}u(n-1)\right\} = \frac{az^{-1}}{1+az^{-1}} \qquad n \geq 1$$

$$\Rightarrow z^{-1}\left\{\frac{az^{-1}}{1+az^{-1}}\right\} = (-1)^{n+1}u(n-1) \ldots\ldots\ldots\ldots(1)$$

Again we know if $Z\{nx(n)\} = -z\frac{d}{dz}X(z)$

thus if $Z^{-1}\left\{-z\frac{d}{dz}X(z)\right\} = y(n)$ and $Z^{-1}\{X(z)\} = \frac{y(n)}{n}$

Hence $Z^{-1}\left\{\left(1+az^{-1}\right)\right\} = \frac{(-1)^{n+1}a^n u(n-1)}{n}$

Problem 2.5: Find the inverse z-transform of $\left(1-2z^{-1}+z^{-2}\right)\left(1+2z^{-1}+4z^{-2}+8z^{-3}+16z^{-4}\right)$

Solution:

We have the convolution theorem as:

If $Z^{-1}\{X_1(z)\} = x_1(n)$ and $Z^{-1}\{X_2(z)\} = x_2(n)$

then $Z^{-1}\{X_1(z)X_2(z)\} = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)$

Here $Z^{-1}\left\{1-2z^{2}-z^{-2}\right\} = \{1,-2,1\}$

and $Z^{-1}\left\{1+2z^{-1}+4z^{-2}+8z^{-3}+16z^{-4}\right\} = \left\{2^n\right\}$ where $0 \leq n \leq 4$

$$\therefore \quad Z^{-1}\left\{\left(1-2z^{-1}+z^{-2}\right)\left(1+2z^{-1}+4z^{-2}+8z^{-3}+16z^{-4}\right)\right\}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)$$

where $x_1(m) = \{1,-2,1\}$ and $x_2(m) = \left\{2^m\right\}$, $0 \leq m \leq 4$

$x_1(i)=0$, $i<0$, $x_1(0)=1$, $x_1(1)=-2$, $x_1(2)=1$, $x_1(k)=0;\ k>2$

and

$x_2(i)=0, i<0$, $x_2(0)=1$, $x_2(1)=2$, $x_2(2)=4$, $x_2(3)=8$, $x_2(4)=16$, $x_2(j)=0;\quad j>4$.

So we will have to calculate terms

$x_1(0)x_2(0)$; $x_1(0)x_2(1)+x_1(1)x_2(0)$; $x_1(0)x_2(2)+x_1(1)x_2(1)+x_1(2)x_2(0)$;

$x_1(0)x_2(3)+x_1(1)x_2(2)+x_1(2)x_2(1)$; $x_1(0)x_2(4)+x_1(1)x_2(3)+x_1(2)x_2(2)$;

$x_1(1)x_2(4)+x_1(2)x_2(3)+x_1(2)x_2(2)$ and $x_1(2)x_2(4)$

And the values are $1,\ 2-2,\ 4-4+1,\ 8-8+2,\ 16-16+4,\ -32+8,\ 16$

i.e., $\{1,0,1,2,4,-24,16\}$

Thus we have the required inverse transform as

$$\{1,0,1,2,4,-24,16\}$$

Note: The result can also be obtained by the following way

$$\left(1-2z^{-1}+z^{-2}\right)\left(1+2z^{-1}+4z^{-2}+8z^{-3}+16z^{-4}\right)=1+z^{-2}+2z^{-3}+4z^{-4}-24z^{-5}+16z^{-6}$$

which is nothing but the z-transform of $\{1,0,1,2,4,-24,16\}$

Problem 2.6: Find the inverse z-transform of $\dfrac{1}{\left(1-az^{-1}\right)^2}$

Solution:

We know, $Z\left\{\dfrac{1}{1-az^{-1}}\right\}=a^n u(n)$

and the convolution theorem as $Z^{-1}\left\{X_1(z)X_2(z)\right\}=\sum_{m=-\infty}^{\infty}x_1(m)x_2(n-m)$

where $Z^{-1}\left\{X_1(z)\right\}=x_1(n)$ and $Z^{-1}\left\{X_2(z)\right\}=x_2(n)$

Let $X_1(z)=\dfrac{1}{1-az^{-1}}=X_2(z)$ then $x_1(m)=a^m u(n)$ and $x_2(n-m)=a^{n-m}u(n-m)$

$$\therefore \quad Z^{-1}\left\{\frac{1}{\left(1-az^{-1}\right)^2}\right\}=\sum_{m=-\infty}^{\infty} a^m u(n)\, a^{n-m} u(n-m)$$

$$=\sum_{m=0}^{n} a^m a^{n-m} = a^n \sum_{m=0}^{n} 1 = (n+1)a^n$$

Problem 2.7: Solve the difference equation using Z-transform

$y(n+2)-4y(n+1)+3y(n)=5^n$; given $y(0)=1$, $y(1)=1$

Solution: Given that,

$$y(n+2)-4y(n+1)+3y(n)=5^n \;;\; y(0)=1 \;,\; y(1)=1$$

Let $Z\{y(n)\}=Y(z)$

Taking Z-transform on the both sides of the given equation we get

$$Z\{y(n+2)\}-4Z\{y(n+1)\}+3Z\{y(n)\}=Z\{5^n\}$$

$$\text{or, } Z^2\left(Y(z)-y(0)-\frac{y(1)}{z}\right)-4Z\left(Y(z)-y(0)\right)+3Y(z)=\frac{1}{1-5z^{-1}}$$

$$\text{or, } Y(z)\left(z^2-4z+3\right)=\frac{z}{z-5}+z^2-3z$$

$$\text{or, } Y(z)=\frac{z}{(z-3)(z-1)(z-5)}+\frac{z(z-3)}{(z-3)(z-1)}$$

$$=\frac{z\{1+(z-3)(z-5)\}}{(z-1)(z-3)(z-5)}$$

$$\text{i.e., } \frac{Y(z)}{z}=\frac{1+(z-3)(z-5)}{(z-1)(z-3)(z-5)}$$

$$= \frac{1}{z-1} + \frac{1}{(z-1)(z-3)(z-5)}$$

$$= \frac{1}{z-1} + \frac{1}{(z-1)(1-3)(1-5)} + \frac{1}{(3-1)(z-3)(3-5)} + \frac{1}{(5-1)(5-3)(z-5)}$$

$$= \frac{1}{z-1} + \frac{1/8}{z-1} - \frac{1/4}{z-3} + \frac{1/8}{z-5}$$

$$= \frac{9/8}{z-1} - \frac{1/4}{z-3} + \frac{1/8}{z-5}$$

i.e., $Y(z) = \frac{9}{8}\,\frac{1}{1-1/z} - \frac{1}{4}\,\frac{1}{1-3/z} + \frac{1}{8}\,\frac{1}{1-5/z}$

Taking inverse z-transform we get

$$y(n) = \frac{9}{8}1^n - \frac{1}{4}3^n + \frac{1}{8}5^n$$

i.e., $y(n) = \frac{9}{8} - \frac{1}{4}3^n + \frac{1}{8}5^n$

CHAPTER-3

DISCRETE FOURIER TRANSFORM AND FAST FOURIER TEANSFORM

Frequency analysis of discrete-time signals is usually and most conveniently performed on a digital signal processor, which may be a general-purpose digital computer or specially designed digital hardware. To perform frequency analysis on a discrete-time signal$\{x(n)\}$, we convert the time-domain sequence to an equivalent frequency-domain representation. We know that such a representation is given by the Fourier transform $X(\omega)$ of the sequence $\{X(n)\}$. However, $X(\omega)$ is a continuous function of frequency and therefore, it is not a computationally convenient representation of the sequence $\{X(n)\}$. The discrete Fourier transform (DFT) and inverse discrete Fourier transform (IDFT) are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering. The importance of the DFT and IDFT in such practical applications is due to a large extent on the existence of computationally efficient algorithms, known collectively as fast Fourier transform (FFT) algorithms, for computing the DFT and IDFT. For the sake of quick understanding to the engineers, in this chapter $\sqrt{-1}$ will be represented by *j*, though generally we represent that by *i*. Before discussing about DFT and others we will present some related topics first.

The Fourier Transform

Recall that a periodic signal $x_p(t)$ with periodic T and its exponential Fourier series coefficients $X[k]$ are related by

$$x_p(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\text{®}kf_0t} \qquad X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) e^{-j2\text{®}kf_0t} dt \qquad (3.1)$$

If the period T of a periodic signal $x_p(t)$ is stretched without limit, the periodic signal no longer remains periodic but becomes a single pulse $x(t)$ corresponding to one period of $x_p(t)$ The harmonic spacing $f_0 = 1/T$ approaches zero, and its Fourier series spectrum becomes a continuous curve. In fact, if we replace f_0 by an infinitesimally small quantity $df \to 0$ the discrete frequency kf_0 may be replaced by the continuous frequency f. The factor $1/T$ in the integral relation means that the coefficients $X[k]$ approach zero and are no longer a useful indicator of the spectral content of the aperiodic signal $x(t)$. However, if we eliminate the dependence of $X[k]$ on the offending factor $1/T$ in the integral and work with $TX[k]$ as follows,

$$TX[k] = \int_{-T/2}^{T/2} x_p(t) e^{-j2®kf_0 t} dt$$

the integral on the right-hand side often exists as $T \to \infty$ (even though $TX[k]$ is in indeterminate form), and we obtain meaningful results. Further, since $k f_0 \to f$, the integral describes a function of f. As a result, we define $TX[k]$ and obtain

$$X(f) = \lim_{T \to \infty} TX[k] = \int_{-\infty}^{\infty} x(t) e^{-j2® ft} dt$$

This relation describes the **Fourier transform** $X(f)$ of the signal $x(t)$ and may also be written in terms of the frequency variable μ as

$$X(\mu) = \int_{-\infty}^{\infty} x(t) e^{-j\mu t} dt \quad \text{(the } \mu \text{-form)}$$

The Fourier transform provides a frequency-domain representation of the aperiodic signal $x(t)$.

The Inverse Fourier Transform

A periodic signal $x_p(t)$ can be reconstructed from its spectral coefficients $X[k]$, using

$$x_p(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2®kf_0 t}$$

If $T \to \infty$, resulting in the aperiodic signal $x(t)$, It is quantity $TX[k]$ that describe its spectrum $X(f)$, and we must modify the above expression (multiply and divided by T) to give

$$x_p(t) = \sum_{k=-\infty}^{\infty} TX[k] e^{j2®kf_0 t} \frac{1}{T} = \sum_{k=-\infty}^{\infty} TX[k] e^{j2®kf_0 t} f_0$$

As $T \to \infty$ and $kf_0 \to f$, the summation tends to an integration over $(-\infty, \infty)$. With $f_0 \to df \to 0$, we obtain $$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

This is the inverse Fourier transform, which allows us to obtain $x(t)$ from its spectrum $X(f)$. The inverse transform relation may be also be written in terms of the variable ω (by nothing that $d\omega = 2\pi df$) to give $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad \text{(from the } \omega \text{-form)}$$

Ideal Sampling

Ideal sampling describe a sampled signal as a weighted sum of impulses, the weights being equal to the values of the analog signal at the impulse locations. An ideally sampled signal $x_I(t)$ may be regarded as the product of an analog signal $x(t)$ and a periodic impulse train $i(t)$.

The ideally sampled signal may be mathematically described as

$$x_I(t) = x(t)i(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nt_s) = \sum_{n=-\infty}^{\infty} x(nt_s)\delta(t - nt_s) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nt_s)$$

Here the discrete signal $x[n]$ simply represents the sequence of sample values $x(nt_s)$. Clearly, the sampling operation leads to a potential loss of information in the ideally sampled signal $x_I(t)$, when compared with its underlying analog counterpart $x(t)$. The smaller the sampling interval t_s, the less is the loss of information.

Intuitively, there must always be some loss of information, no matter how small an interval we use. Fortunately, our intuition notwithstanding, it is indeed possible to sample signals without any loss of information.

Let us consider a signal $x(t)$, which is band-limited to some finite frequency B .Let the impulse train $i(t)$ is a periodic signal with period $T = t_s = 1/S$ and Fourier series coefficients $I(k) = S$. Its Fourier transform is a train of impulses (at $f = kS$) whose strengths equal $I(k)$

$$I(f) = \sum_{k=-\infty}^{\infty} I[k]\delta(f - kS) = S\sum_{k=-\infty}^{\infty} \delta(f - kS)$$

The ideally sampled signal $x_I(t)$ is the product of $x(t)$ and $i(t)$. Its spectrum $X_I(f)$ is thus described by the convolution

$$X_I(f) = X(f) * I(f) = X(f) * S\sum_{k=-\infty}^{\infty} ¢\,(f - kS) = S\sum_{k=-\infty}^{\infty} X(f - kS)$$

The spectrum $X_I(f)$ consists of $X(f)$ and its shifted replicas or images. It is periodic in frequency, with a period that equals the sampling rate S.

Since the spectral image at the origin extends over $(-B, B)$, and the next image (centered at S) extends over $(S - B, S + B)$, the image will not overlap if

$S - B > B$ or $S > 2B$

There will have three choices of the sampling frequency S of the spectra of an ideally sampled band-limited signal. As long as the images do not overlap, each period is a replica of the scaled analog signal spectrum $SX(f)$. We can thus extract $X(f)$ (and hence $x(t)$ as the principal period of $X_I(f)$ (between $-0.5S$ and $0.5S$). By passing the ideally sampled signal through an ideal lowpass filter with a cutoff frequency of $0.5S$ and a gain of $1/S$ over the frequency range $-0.5S \le f \le 0.5S$

The **sampling theorem** tells us that an analog signal band-limited to a frequency B can be sampled without loss of information if the sampling rate S exceeds 2B (or the sampling interval t_s is smaller than $\frac{1}{2B}$). The critical sampling rate $S_N = 2B$ is often called the **Nyquist rate** or **Nyquist frequency** and the critical sampling interval $t_N = 1/S_N = 1/2B$ is called the **Nyquist interval**.

If the sampling rate S is less than $2B$, the spectral images overlap and the principle period $(-0.5S, 0.5S)$ of $X_I(f)$ is no longer an exact replica of $X(f)$. In this case, we cannot exactly recover $x(t)$, and there is loss of information due to **undersampling**. Undersampling results in spectral overlap. Components of $X(f)$ outside the principle range $(-0.5S, 0.5S)$ fold back into this range (due to the spectral overlap from adjacent images). Thus, frequencies higher than $0.5S$ appear as lower frequencies in the principal period. This is aliasing. The frequency $0.5S$ is also called the **folding frequency.**

Aliasing: A frequency $|f_0| > 0.5S$ gets aliased to a lower frequency f_a in the range $(-0.5S,\, 0.5S)$.

Sampling is a band-limiting operation in the sense that in practice we typically extract only the principal period of the spectrum, which is band-limited to the frequency range $(-0.5S, 0.5S)$. Thus, the highest frequency we can recover or identify is $0.5S$ and depends only on the sampling rate S .

Sampling of Sinusoids and Periodic Signals

The Nyquist frequency for a sinusoid $x(t) = \cos(2\pi f_0 t + \theta)$ is $S_N = 2f_0$. The Nyquist interval is $t_N = 1/2f_0$, or $t_N = T/2$. This amounts to taking more than two samples per period. If, for example, we acquire just two samples per period, starting at a zero crossing, all sample values will be zero, and will yield no information.

If a signal $x(t) = \cos(2\pi f_0 t + \theta)$ is sampled at S , the sampled signal is $x[n] = \cos(2\pi f_0 / S + \theta)$. Its spectrum is periodic, with principle period $(-0.5S,\, 0.5S)$. If $f_0 < 0.5S$, there is no aliasing, and the principle period shows a pair of impulses at $\pm f_0$ (with strength 0.5). If $f_0 > 0.5S$, we have aliasing. The components at $\pm f_0$ are aliased to a lower frequency $\pm f_a$ in the principle range. To find the aliased frequency $|f_a|$, we subtract integer multiples of the sampling frequency from f_0 unit the result $f_a = f_0 - NS$ lies in the principle range $(-0.5S,\, 0.5S)$. The spectrum then describes a sampled version of the lower-frequency aliased signal $x_a(t) = \cos(2\pi f_0 t + \theta)$. The aliased frequency always lies in the principle range.

A periodic signal $x_p(t)$ with period T can be described by a sum of sinusoids at the fundamental frequency $f_0 = 1/T$ and its harmonics $k\, f_0$. In general, such a signal not be band-limited and cannot be sampled without aliasing for any choice of sampling rate.

The spectrum of a sampled signal is not only continuous but also periodic. The periodicity is a consequence of the duality and reciprocity between time and frequency and leads to the formulation of the discrete-time Fourier transform (DTFT).

3.1 The Discrete-Time Fourier Transform

The Discrete-Time Fourier Transform describe the spectrum of discrete-time signals and formalizes the concept that discrete-time signals have periodic spectra. Ideal sampling of an analog signal $x(t)$ leads to the ideally sampled signal $x_I(t)$ whose spectrum $X_p(f)$ is periodic. We have

$$x_I(t) = \sum_{k=-\infty}^{\infty} x(kt_s)\delta(t-kt_s) \qquad X_p(f) = S\sum_{k=-\infty}^{\infty} X(f-kS) \qquad (3.1.1)$$

Using the Fourier transform pair $\delta(t-\tau) \Leftrightarrow \exp(-j2\pi\tau f)$, the spectrum $X_p(f)$ may also be described by

$$x_I(t) = \sum_{k=-\infty}^{\infty} x(kt_s)\delta(t-kt_s) \qquad X_p(f) = \sum_{k=-\infty}^{\infty} x(kt_s)e^{-j2\pi kt_s f} \qquad (3.1.2)$$

Note that $X_p(f)$ is periodic with period S and its central $SX(f)$. To recover the analog signal $x(t)$, we passed the sampled signal through an ideal lowpass filter whose gain equals $1/S$ over $-0.5S \le f \le 0.5S$.

Formally, we obtain $x(t)$ (or its samples $x(nt_s)$) from the inverse Fourier transform result

$$x(t) = \frac{1}{S}\int_{-S/2}^{S/2} X_p(f)e^{j2\pi ft}df \qquad x(nt_s) = \frac{1}{S}\int_{-S/2}^{S/2} X_p(f)e^{j2\pi ft_s}df \qquad (3.1.3)$$

Equations (3.1.2) and (3.1.3) define a transform pair. They allow us to obtain the periodic spectrum $X_p(f)$ of an ideally sampled signal from its samples $x(nt_s)$, and to recover the samples $x(nt_s)$ from the spectrum. We point out that these relations are the exact duals of the Fourier series relations for a periodic signal $x_p(t)$ and its discrete spectrum $X[k]$ (the Fourier series coefficients). We can revise these relations for discrete-time signals if we use the digital frequency $F = f/S$ and replace $x(nt_s)$ by the discrete sequence $x[n]$ to obtain

$$X_p(F) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi kF} \qquad x[n] = \frac{1}{S}\int_{-1/2}^{1/2} X_p(F)e^{j2\pi nF}dF \quad \text{(the F-form)} \qquad (3.1.4)$$

The first result define $X_p(F)$ as the discrete-time Fourier transform (DTFT) of $x[n]$. The second result is the inverse DTFT (IDTFT), which allows us to recover $x[n]$ from its spectrum. The DTFT $X_p(F)$ is periodic with unit period because it assumes unit spacing samples of $x[n]$. The interval $-0.5 \le F \le 0.5$ (or $0 \le F \le 1$) defines the **principle period**.

The DTFT relations may also be written in terms of the radian frequency Ω as

$$X_p(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-jk\Omega} \qquad x[n] = \frac{1}{2®} \int_{-®}^{®} X_p(\Omega) e^{jn\Omega} d\Omega \qquad (3.1.5)$$

The quantity $X_p(\Omega)$ is now periodic with period $\Omega = 2®$ and represents a scaled (stretched by $2®$) version of $X_p(F)$. The principle period of $X_p(\Omega)$ corresponds to the interval $-® \le \Omega \le ®$ or $0 \le \Omega \le 2®$. We will find it convenient to work with the F-form because, as in the case of Fourier transforms, it rids us of factors of $2®$ in many situations.

3.1.1 Connection between the DTFT and the Fourier Transform

If the signal $x[n]$ is obtained by ideal sampling of an analog signal $x(t)$ at a sampling rate S , the Fourier transform $X_I(F)$ of the analog impulse train $x_I(t) = \sum x(t) ¢ (t - k/S)$ equals $X_P(F)\big|_{F \to f/S}$ and represents a frequency-scaled version of $X_p(F)$ with principle period $(-0.5S, 0.5S)$. If S exceeds the Nyquist sampling rate, the Fourier transform $X(f)$ of $x(t)$ equals the principle period of $SX_P(F)\big|_{F \to f/S}$. If S is below the Nyquist rate, $SX_p(F)$ matches the periodic extension of $X(f)$. In other words, the DTFT of a discrete-time signal $x[n]$ is related to both the Fourier transform $X_I(F)$ of the underlying impulse-sampled analog signal $x_I(t)$ and to the Fourier transform $X(f)$ of $x(t)$.

3.1.3 The DFS and the DFT

Sampling and duality provide the basis for the connection between all of the frequency-domain transforms and the concepts are worth repeating. Sampling in one domain induces a periodic extension in the other. The sample spacing in one domain is the reciprocal of the period in the other. Period analog signals have discrete spectra, and discrete-time signals have continuous periodic spectra. A consequence of these concepts is that a sequence that is both discrete and periodic in one domain is also discrete and periodic in the other. This leads to the development of the **discrete Fourier transform** (DFT) and **discrete Fourier series** (DFS), allowing us a practical means of arriving at the sampled spectrum of sampled signals using digital computers. The connections between the various transforms are summarized in Table 3.1

Table 3.1 Connections Between Various Transforms

Operation in the Time Domain	**Result in the Frequency Domain**	**Transform**
Aperiodic continuous $x(t)$	Aperiodic continuous $X(f)$	FT
Periodic extension of $x(t) \Rightarrow x_p(t)$ Period=T	Sampling of $X(f) \Rightarrow X[k]$ Sampling interval $=1/T = f_0$	FS
Sampling of $x_p(t) \Rightarrow x_p[n]$ Sampling interval $= t_s$	Periodic extension of $X[k] \Rightarrow X_{DFS}[k]$ Period$= S = 1/t_s$	DFS
Sampling of $x(t) \Rightarrow x[n]$ Sampling interval =1	Periodic extension of $X(f) \Rightarrow X_p(F)$ Period=1	DTFT
Periodic extension of $x[n] \Rightarrow x_p[n]$ Period=N	Sampling of $X_p[F] \Rightarrow X_{DFT}[k]$ Sampling interval $=1/N$	DFT

3.2 DFT and IDFT

The *N*-point discrete Fourier transform (DFT) $X_{DFT}[k]$ of an N-sample signal $x[n]$ and the inverse Fourier transform (IDFT), which transforms $X_{DFT}[k]$ to $x[n]$, are defined by

$$X_{DFT}(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2®nk/N} \qquad k = 0,1,2,\ldots,N-1 \qquad (3.2.1)$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_{DFT}(k)\, e^{j2®nk/N} \qquad n = 0,1,2,\ldots,N-1 \qquad (3.2.2)$$

Each relation is a set of *N* equations. Each DFT sample is found as a weighted sum of all the sample in $x(n)$. One of the most important properties of the DFT and its inverse in implied periodicity. The exponential $\exp(\pm j2®nk/N)$ in the defining relations is periodic in both *n* and *k* with period *N*:

$$e^{j2®nk/N} = e^{j2®(n+N)k/N} = e^{j2®n(k+N)/N}$$

As a result, the DFT and its inverse are also periodic with period N, and its sufficient to compute the results for only one period (0 to N-1). Both $x[n]$ and $X_{DFT}[k]$ have a starting index of zero.

Let us give an example to calculate the DFT

Let $x(n)=\{1,2,1,0\}$. with $N=4$, and $e^{-j2®nk/N} = e^{-j2®nk/2}$, we successively compute

$$k=0: \quad X_{DFT}(n)=\sum_{n=0}^{3}x(n)e^{0} = 1+2+1+0=4$$

$$k=1: \quad X_{DFT}(1)=\sum_{n=0}^{3}x(\mathrm{n})e^{-jn®/2} = 1+2e^{-j®/2}+e^{-j®}+0=-j2$$

$$k=2: \quad X_{DFT}(2)=\sum_{n=0}^{3}x(2)e^{-jn®} = 1+2e^{-j®}+e^{-j2®}+0=0$$

$$k=3: \quad X_{DFT}(3)=\sum_{n=0}^{3}x(\mathrm{n})e^{-j3n®/2} = 1+2e^{-j3®/2}+e^{-j3®}+0=j2$$

The DFT is thus $X_{DFT}(\mathrm{k})=\{4,-j2,0,\mathrm{j}2\}$.

Here is an example to calculate the DFT of a sequence and get that back through IDFT.

Let us consider a sequence $x(n)=\{2,0,1,-2\}$. The DFT in this case will be given by $X(l)=\sum_{k=0}^{3}x(k)e^{-j2®lk/4}$, where $l=0,1,2,3$

Hence $X(0)=\sum_{k=0}^{3}x(k)e^{-j®.0.k/2} = \sum_{k=0}^{3}x(k)=1$

$$X(1)=\sum_{k=0}^{3}x(k)e^{-j®.1.k/2} = \sum_{k=0}^{3}x(k)\left[\cos(k®/2)-j\sin(k®/2)\right]=2.1+0+1.(-1)+(-2).(j)=1-2\,\mathrm{j}$$

$$X(2)=\sum_{k=0}^{3}x(k)e^{-j®.2.k/2} = \sum_{k=0}^{3}x(k)\left[\cos(k®)-j\sin(k®)\right]=2.1+0+1.(1)+(-2).(-1)=5$$

$$X(3)=\sum_{k=0}^{3}x(k)e^{-j®.3.k/2} = \sum_{k=0}^{3}x(k)\left[\cos(3k®/2)-j\sin(3k®/2)\right]$$

$$=2.1+0+1.(-1)+(-2).(-j)=1+2\,\mathrm{j}$$

From the definition we will have

$$x(k)=\frac{1}{4}\sum_{l=0}^{3}X(l)e^{j2®lk/4} \text{, where } k=0,1,2,3$$

Thus

$$x(0)=\frac{1}{4}\sum_{l=0}^{3}X(l)e^{j®.0.l/2} = \frac{1}{4}(1+1-2j+5+1+2j)=2$$

$$x(1)=\frac{1}{4}\sum_{l=0}^{3}X(l)e^{j®.1.l/2}=\frac{1}{4}\sum_{l=0}^{3}X(l)\left[\cos(l® / 2)+j\sin(l® / 2)\right]$$

$$=\frac{1}{4}\left[1.1+(1-2j).j+5.(-1)+(1+2j).(-j)\right]=0$$

$$x(2)=\frac{1}{4}\sum_{l=0}^{3}X(l)e^{j®.2.l/2}=\frac{1}{4}\sum_{l=0}^{3}X(l)\left[\cos(l®)+j\sin(l®)\right]$$

$$=\frac{1}{4}\left[1.1+(1-2j).(-1)+5.(1)+(1+2j).(-1)\right]=1$$

$$x(3)=\frac{1}{4}\sum_{l=0}^{3}X(l)e^{j®.3.l/2}=\frac{1}{4}\sum_{l=0}^{3}X(l)\left[\cos(3l® / 2)+j\sin(3l® / 2)\right]$$

$$=\frac{1}{4}\left[1.1+(1-2j).(-j)+5.(-1)+(1+2j).(j)\right]=-2$$

Now we will illustrate the properties of DFT in the following with examples:

(a) Let $y(n)=\{1,2,3,4,5,0,0,0\}$, $n=0,1,2,...,7.$ Find one period of the circularly shifted signals $f(n)=y(n-2)$, $g(n)=y(n+2)$, and the circularly folded signal $h(n)=y(-n)$ over $0\le n\le 7$.

1. To create $f(n)=y(n-2)$, we move the last two samples to the beginning. So,

$f(n)=y(n-2)=\{0,0,1,2,3,4,5,0\}$, $n=0,1,2,...,7.$

2. To create, $g(n)=y(n+2)$ we move the first two samples to the end. So,

$g(n)=y(n+2)=\{3,4,5,0,0,0,1,2\}$, $n=0,1,2,...,7.$

3. To create, $h(n)=y(-n)$ we fold $y(n)$ to $\{0,0,0,5,4,3,2,1\}$, $n=-7,-6,-5,...,0$ and create its periodic extension by moving all samples (except $y(0)$) past $y(0)$ to get

$h(n)=y(-n)=\{1,0,0,0,5,4,3,2\}$, $n=0,1,2,...,7.$

(b) Let us find the DFT of $x(n)=\{1,1,0,0,0,0,0,0\}$, $n=0,1,2,...,7.$

Since only $x(0)$ and $x(1)$ are nonzero, the upper index in the DFT summation will be n=1 and the DFT reduces to

$$X_{DFT}(k)=\sum_{n=0}^{1}x(n)e^{-j2®nk/8}=1+e^{-j®k/4}\ ,\quad k=0,1,2,...,7.$$

Since $N = 8$, we need compute $X_{DFT}(k)$ only for $k \le 0.5N = 4$. Now, $X_{DFT}(0) = 1 + 1 = 2$ and $X_{DFT}(0) = 1 - 1 = 0$. For the rest $(k = 1, 2, 3)$, we compute,

$X_{DFT}(1) = 1 + e^{-j2®/4} = 1.707 - j0.707$, $X_{DFT}(2) = 1 + e^{-j®/2} = 1 - j$,

$X_{DFT}(3) = 1 + e^{-j3®/4} = 0.293 - j0.707$.

By conjugate symmetry, $X_{DFT}(k) = X^*_{DFT}(N-k) = X^*_{DFT}(8-k)$. This gives

$X_{DFT}(5) = X^*_{DFT}(3) = 0.293 + j0.707$, $X_{DFT}(6) = X^*_{DFT}(2) = 1 + j$,

$X_{DFT}(7) = X^*_{DFT}(1) = 1.707 + j0.707$.

Thus, $X_{DFT}(k) = \{2, 1.707 - j0.707, 0.293 - j0.707, 0, 1 + j, 0.293 + j0.707, 1.707 + j0.707\}$.

(c) Consider the DFT pair $x(n) = \{1, 2, 1, 0\} \Leftrightarrow X_{DFT}(k) = \{4, -j2, 0, \text{j}2\}$ with $N = 4$.

1. (Time Shift) To find $y(n) = x(n-2)$, we move the last two samples to the beginning to get

$y(n) = x(n-2) = \{1, 0, 1, 2\}$, $n = 0, 1, 2, 3.$

To find the DFT of $y(n) = x(n-2)$, we use the time-shift property (with $n_0 = 2$) to give

$Y_{DFT}(k) = X_{DFT}(k) e^{-j2® k n_0/4} = X_{DFT}(k) e^{-jk®} = \{4, \text{j}2, 0, -j2\}$.

2. (Modulation) The sequence $Z_{DFT}(k) = X_{DFT}(k-1)$ equals $\{j2, 4, -j2, 0\}$. Its IDFT is

$z(n) = x(n) e^{j2®n/4} = x(n) e^{j®n/2} = \{1, \text{j}2, -1, 0\}$.

3. (Folding) The sequence $g(n) = x(-n)$ is

$g(n) = \{x(0), x(-1), x(-2), x(-3)\} = \{1, 0, 1, 2\}$

Its IDFT equals to $G_{DFT}(k) = X_{DFT}(-k) = X^*_{DFT}(k) = \{4, \text{j}2, 0, -j2\}$

4. (Conjugation) The sequence $p(n) = x^*(n)$ is $p(\text{n}) = x^*(n) = x(n) = \{1, 2, 1, 0\}$. Its DFT is

$P_{DFT}(k) = X^*_{DFT}(-k) = \{4, \text{j}2, 0, -j2\}^* = \{4, -j2, 0, \text{j}2\}$.

5. (Product) The sequence $h(n) = x(n)x(n)$ is the point wise product. So, $h(n) = \{1, 4, 1, 0\}$.

Its DFT is $H_{DFT}(k) = \dfrac{1}{4} X_{DFT}(k) \sqcup X_{DFT}(k) = \dfrac{1}{4}\{4, -j2, 0, \text{j}2\} \sqcup \{4, \text{j}2, 0, -j2\}$

We need to keep in mind that this is a periodic convolution.

The result is $H_{DFT}(k) = \dfrac{1}{4}\{24, -j16, 0, \text{j}16\} = \{6, -j4, 0, \text{j}4\}$

6. (Periodic Convolution) The period convolution is $c(n) = x(n) \sqcup x(n)$ gives

$c(n) = \{1, 2, 1, 0\} \sqcup \{1, 2, 1, 0\} = \{2, 4, 6, 4\}$

Its DFT is given by point wise product

$C_{DFT}(k) = X_{DFT}(k) X_{DFT}(k) = \{16, -4, 0, -4\}$.

7. (Regular Convolution) The regular convolution $s(n) = x(n) * x(n)$ gives

$s(n) = \{1, 2, 1, 0\} \sqcup \{1, 2, 1, 0\} = \{1, 4, 6, 4, 1, 0, 0\}$.

Since $x(n)$ has 4 samples $(N = 4)$, the DFT $S_{DFT}(k)$ of $s(n)$ is the product of the DFT of the zero-padded (to length $N + N - 1 = 7$) signal $x_z(n) = \{1, 2, 1, 0, 0, 0, 0\}$ and equals

$\{16, -2.34 - j10.28, -2.18 + j1.05, 0.02 + j0.03, 0.02 - j0.03, -2.18 - j1.05, -2.35 + j10.28\}$

8. **(Central Ordinates)** It is easy to check that $x(0) = \dfrac{1}{4}\sum X_{DFT}(k)$ and $X_{DFT}(0) = \sum x(n)$.

9. (Perseval's Relation) We have $\sum \left| x(n)^2 \right| = 1 + 4 + 1 + 0 = 6$.

Since $X^2_{DFT}(k) = \{16, -4, 0, 4\}$; we also have $\dfrac{1}{4}\sum \left| X_{DFT}(k) \right|^2 = \dfrac{1}{4}(16 + 4 + 4) = 6$.

3.2.1 The DFT of Periodic Signals and the DFS

The Fourier series relations for a periodic signal $x_p(t)$ are

$$x_p(t) = \sum_{k=-\infty}^{\infty} X(k) e^{j2\circledR kf_0t} \qquad X(k) = \frac{1}{T}\int_T x_p(t) e^{-j2\circledR kf_0t} dt \qquad (3.2.3)$$

If we acquire $x(n)$, $n = 0, 1, ..., N - 1$ as N samples of $x_p(t)$ over one period using a sampling rate of S Hz (corresponding to a sampling interval of t_s) and approximate the integral expression for $X(k)$ by a summation using $dt \to t_s$, $t \to nt_s$, $T = Nt_s$, and $f_0 = \dfrac{1}{T} = \dfrac{1}{Nt_s}$, we obtain

$$X_{DFS}(k) = \frac{1}{Nt_s}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kf_0 t_s} t_s = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \quad , \qquad k = 0,1,\ldots, N-1 \qquad (3.2.4)$$

The quantity $X_{DFS}(k)$ defines the **discrete Fourier series** (DFS) as an approximation to the Fourier series coefficients of a periodic signal and equals *N* times the DFT.

3.2.2 The Inverse DFS

To recover $x(n)$ from one period of $X_{DFS}(k)$, we use the Fourier series reconstruction whose summation index covers one period (from k=0 to k=N-1) to obtain

$$x(n) = \sum_{k=0}^{N-1} X_{DFS}(k)e^{j2\pi kf_0 nt_s} = \sum_{k=0}^{N-1} X_{DFS}(k)e^{j2\pi nk/N} \quad , \qquad n = 0,1,2,\ldots, N-1 \qquad (3.2.5)$$

This relation describes the inverse discrete Fourier series (IDFS). The sampling interval t_s does not enter into the computation of the DFS or its inverse. Except for a scale factor, the DFS and DFT relations are identical.

Here is an example of the DFT of a Sinusoid:

The signal $x(t) = 4\cos(100\pi t)$ is sampled at twice the Nyquist rate for three full periods. The frequency of $x(t)$ is 50Hz, the Nyquist rate is 100Hz, and the sampling frequency is *S*=200Hz.

The digital frequency is $F = 50/200 = 1/4 = 3/12 = k/N$. This means *N*=12 for three full periods. The two nonzero DFT values will appear at *k*=3 and *k*=*N*-3=9. The nonzero DFT values will be $X(3) = X(9) = (0.5)(4)(N) = 24$

3.3 Matrix representation of DFT

The formulas for the DFT and IDFT given by (3.2.1) and (3.2.2) may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^k \qquad k = 0,1,\ldots,N-1 \qquad (3.3.1)$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-k} \qquad n = 0,1,\ldots,N-1 \qquad (3.3.2)$$

where, by definition, $W_N = e^{-j2\pi/N}$ which is an Nth root of unity.

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and $(N-1)$ complex additions. Hence the N-point DFT values can be computed in a total of…..complex multiplications and $N(N-1)$ complex additions.

Let us define an N-point vector $\boldsymbol{X}_N$of the signal sequence $x(n)$, $n = 0,1,2, \dots \dots \dots, N-1$, an N-point vector $\boldsymbol{X}_N$of frequency samples, and an $N \times N$ matrix $\boldsymbol{W}_N$ as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \boldsymbol{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix},$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \qquad (3.3.3)$$

With these definitions, the N-point DFT may be expressed in matrix form as

$$\boldsymbol{X}_N = \boldsymbol{W}_N \boldsymbol{x}_N \qquad (3.3.4)$$

where $\boldsymbol{W}_N$ Is the matrix of the linear transformation. We observe that $\boldsymbol{W}_N$ is a symmetric matrix. If we assume that the inverse of $\boldsymbol{W}_N$ exists, then (3.3.4) can be inverted by premultiplying both sides by $\boldsymbol{W}_N^{-1}$. Thus we obtain

$$\boldsymbol{x}_N = \boldsymbol{W}_N^{-1} \boldsymbol{X}_N$$

But this is just an expression for the IDFT.

In fact, the IDFT as given by (3.3.2), can be expressed in matrix form as

$$\boldsymbol{x}_N = \frac{1}{N} \boldsymbol{W}_N^* \boldsymbol{X}_N \qquad (3.3.5)$$

where $\boldsymbol{W}_N^*$ denotes the complex conjugate of the matrix $\boldsymbol{W}_N$. Comparison of (3.3.5) with (3.3.4) leads us to conclude that $\boldsymbol{W}_N^{-1} = \frac{1}{N} \boldsymbol{W}_N^*$ (3.3.6)

which, in turn, implies that $\boldsymbol{W}_N \boldsymbol{W}_N^* = N \boldsymbol{I}_N$ (3.3.7) where $\boldsymbol{I}_N$ is an $N \times N$ identity matrix. Therefore, the matrix $\boldsymbol{W}_N$ in the transformation is an orthogonal (unitary) matrix. Furthermore, its inverse exists and is given as $\boldsymbol{W}_N^*/N$. Of course, the existence of the inverse of $\boldsymbol{W}_N$ was established previously from our derivation of the IDFT.

Example 3.3.1 Compute the DFT of the four-point sequence $x(n) = (0 \;\; 1 \;\; 2 \;\; 3)$

Solution: The first step is to determine the matrix $\boldsymbol{W}_4$. By exploiting the periodicity property of $\boldsymbol{W}_4$ and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

The matrix $\boldsymbol{W}_4$ may be expressed as

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\boldsymbol{X}_4 = \boldsymbol{W}_4 \boldsymbol{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\boldsymbol{x}_4$ may be determined by conjugating the elements in $\boldsymbol{W}_4$ to obtain $\boldsymbol{W}_4^*$.and applying the formula (3.3.5).

3.4 Relationship of the DFT to other transforms

3.4.1 Relationship to the z-transform:

Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.4.1}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi \;\; /N}$,0,1,2,…,$N-1$, we obtain

$$X(k) \equiv X(z)|_{z=e^{j2\pi \;\; k/N}} \qquad k = 0,1,\ldots,N-1 \tag{3.4.2}$$
$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \quad /N}$$

The expression in (3.4.2) is identical to the Fourier transform $X(\omega)$ evaluated at the N equally spaced frequencies $\omega_k = 2\pi\ \ /N$, $k = 0,1,\dots,N-1$.

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Consequently, X(z) can be expressed as a function of the DFT $\{X(k)\}$ as follows

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi\ \ /N}\right] z^{-n} \qquad (3.4.3)$$

i.e. $$X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sum_{n=0}^{N-1}\left(e^{j2\pi\ \ /N}z^{-1}\right)^n$$

or, $$X(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j2\pi\ \ /N}z^{-1}}$$

When evaluated on the unit circle, (3.4.3) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1-e^{-j\omega}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{-j(\omega-2\pi\ \ /N)}}$$

This expression for the Fourier transform is a polynomial (Lagrange) interpolation formula for $X(\omega)$ expressed in terms of the values $\{X(k)\}$ of the polynomial at a set of equally spaced discrete frequencies $\omega_k = 2\pi\ \ /N$, $k = 0,1,\dots,N-1$

3.4.2 Relationship to the Fourier series coefficients of a continuous-time signal.

Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a = \sum_{k=-\infty}^{\infty} c_k e^{j2@kF_0}$$

where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_P = 1/T$,we obtain the discrete-time sequence

$$x(n) \equiv x_a(n\ \ ) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\ \ F_0 n} = \sum_{n=-\infty}^{\infty} c_k e^{j2\pi\ \ /N} = \sum_{k=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} c_{k-l}\right] e^{j2\pi\ \ /N}$$

It is clear that the above equation is in the form of an IDFT formula, where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-l} \equiv N\tilde{c}_k$$

and $\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-l}$

Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

For a N-point sequence the convolution theorem takes the following form

If IDFT $\frac{1}{N}\sum_{l=0}^{N-1} X(l)e^{j2®lk/N} = x(k)$ and IDFT $\frac{1}{N}\sum_{l=0}^{N-1} Y(l)e^{j2®lk/N} = y(k)$ then

IDFT $\frac{1}{N}\sum_{l=0}^{N-1} X(l)Y(l)e^{j2®lk/N} = \sum_{m=0}^{N-1} x(m)y(k-m)$, where the right hand side product is known as the convulation. Let us examine with an example how the product is calculated and for the purpose we take two sequence $x(n)=\{4,3,2,1\}$ and $y(n)=\{1,2,3,4\}$, and their product as $z(n)$.

Thus $$z(0)=\sum_{m=0}^{3} x(m)y(-m) = x(0).y(0)+x(1).y(-1)+x(2).y(-2)+x(3).y(-3)$$
$$= 4.1+3.4+2.3+1.2 = 24$$

$$z(1)=\sum_{m=0}^{3} x(m)y(1-m) = x(0).y(1)+x(1).y(0)+x(2).y(-1)+x(3).y(-2)$$
$$= 4.2+3.1+2.4+1.3 = 22$$

$$z(2)=\sum_{m=0}^{3} x(m)y(2-m) = x(0).y(2)+x(1).y(1)+x(2).y(0)+x(3).y(-1)$$
$$= 4.3+3.2+2.1+1.4 = 24$$

and $$z(3)=\sum_{m=0}^{3} x(m)y(3-m) = x(0).y(3)+x(1).y(2)+x(2).y(1)+x(3).y(0)$$
$$= 4.4+3.3+2.2+1.1 = 30$$

3.5 Efficient Computation of the DFT: FFT Algorithms

In view of the importance of the DFT in various digital signal processing applications, such as linear filtering, correlation analysis, and spectrum analysis, its efficient computation is a topic that has received considerable attention by many mathematicians, engineers, and applied scientists. Basically the computational problem for the DFT is to compute the sequence $\{X(k)\}$ of N complex-valued numbers given another sequence of data $\{x(n)\}$ of length N, according to the formula

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^k \qquad 0 \le k \le N-1 \tag{3.5.1}$$

where $$W_N = e^{-j2\pi/N} \tag{3.5.2}$$

In general, the data sequence $x(n)$ is also assumed to be complex valued.

Similarly, the IDFT becomes

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-k} \qquad 0 \le n \le N-1 \tag{3.5.3}$$

Since the DFT and IDFT involves basically the same type of computations, our discussion of the efficient computational algorithms for the DFT applies as well to the efficient computation of the IDFT.

We observe that for each value of k, direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and $N-1$ complex additions ($4N-2$ real additions). Consequently, to compute all N values of the DFT requires N^2 complex multiplications and $N^2 - N$ complex additions.

Direct computation of the DFT is basically inefficient primarily because it does not exploit the symmetry and periodicity properties of the phase factor W_N

In particular, these two properties are:

Symmetry property: $$W_N^{k+N/2} = -W_N^{-k} \tag{3.5.4}$$

Periodicity property: $$W_N^{k+N} = W_N^k \tag{3.5.5}$$

The computationally efficient algorithms described in this section, known collectively as fast Fourier transform (FFT) algorithms, exploit these two basic properties of the phase factor.

3.5.1 Direct Computation of the DFT

For a complex-valued sequence x(n) of N points, the DFT may be expressed as

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi}{N} + x_1(n) \sin \frac{2\pi}{N} \right] \tag{3.5.6}$$

$$X_1(k) = - \sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi}{N} - x_1(n) \cos \frac{2\pi}{N} \right] \tag{3.5.7}$$

The direct computation of (3.5.6) and (3.5.7) requires:

1. $2N^2$ evaluations of trigonometric functions.
2. $4N^2$ real multiplications.
3. $4N(N-1)$ real additions.
4. A number of indexing and addressing operations.

These operations are typical of DFT computational algorithms. These operations in items 2 and 3 result in the DFT values $X_R(k)$ and $X_I(k)$ The indexing and addressing operations are necessary to fetch the data $x(n)$, $0 \le n \le N-1$ and the phase factors and to store the results. The variety of DFT algorithms optimize each of these computational processing in a different way.

3.5.2 Divide-and-conquer approach to computation of the DFT

The development of computationally efficient algorithms for the DFT is made of possible if we adopt a divide-and-conquer approach. These approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a family of computationally efficient algorithms known collectively as FFT algorithms.

To illustrate the basic notions, let us consider the computation of an N-point DFT, where N can be factored as a product of two integers, that is,

$$N = L \tag{3.5.8}$$

The assumption that N is not a prime number is not restrictive, since we can pad any sequence with zeros to ensure a factorization of the form (3.5.8).

Now the sequence $x(n)$, $0 \le n \le N-1$ can be stored in either a one-dimensional array indexed by n or as a two-dimensional array indexed by l and m, where $0 \le l \le L-1$ and $0 \le m \le M$. Note that l is the row index and m is the column index. Thus, the sequence

$x(n)$ can be stored in a rectangular array in a variety of ways, each of which depends on the mapping of index n to the indies (l, m).

For example, suppose that we select the mapping $\quad n = M \ \ + m \qquad$ (3.5.9)

This leads to an arrangement in which the first row consists of the first M elements of $x(n)$, the second row consists of the next M elements of $x(n)$, and so on. On the other hand, the mapping $\quad n = l + m \qquad$ (3.5.10)

stores the first L elements of $x(n)$ in the first column.

A similar arrangement can be used to store the computed DFT values. In particular, the mapping is from the index k to a pair of indices (p, q), where $0 \leq p \leq L - 1$ and $0 \leq q \leq m - 1$. If we select the mapping $\quad k = M \ \ + q \qquad$ (3.5.11)

the DFT is stored on a row-wise basis, where the first row contains the first M elements of the DFT $X(k)$, the second row contains the next set of M elements, and so on. On the other hand, the mapping $\quad k = q \ \ + p \qquad$ (3.5.12)

result in a column-wise storage of $X(k)$, where the first L elements are stored in the first column, the second set of L elements are stored in the second column, and so on.

Now suppose that $x(n)$ is mapped into the rectangular array $x(l, m)$ and $X(k)$ is mapped into a corresponding rectangular array $X(p, q)$. Then the DFT can be expressed as a double sum over the elements of the rectangular array multiplied by the corresponding phase factors. To be specific, let us adopt a column-wise mapping for $x(n)$ given by (3.5.10) and the row-wise mapping for the DFT given by (3.5.11). Then

$$X(p, q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l, m) W_N^{(M \ \ +q)(m \ \ +l)} \qquad (3.5.13)$$

But $\quad W_N^{(M \ \ +q)(m \ \ +l)} = W_N^{M} \quad W_N^{M} \quad W_N^{M} \quad W_N^{l} \qquad$ (3.5.14)

However, $W_N^{N} \quad = 1$, $W_N^{m} \quad = W_{N/L}^{m} = W_M^{m}$, and $W_N^{M} \quad = W_{N/M}^{p} = W_L^{p}$

With these simplifications, (3.5.13) can be expressed as

$$X(p,q) = \sum_{l=0}^{L-1} \left\{ W_N^{lq} \left[\sum_{m=0}^{M-1} x(l,m)\, W_M^{m} \right] \right\} W_L^{lp} \qquad (3.5.15)$$

The expression in (3.5.15) involves the computation of DFTs of length M and length L. To elaborate, let us subdivide the computation into three steps:

1. First, we compute the M-point DFTs

$$F(l,q) = \sum_{m=0}^{M-1} x(l,m)\, W_M^{m}\ , \quad 0 \le q \le M-1 \qquad (3.5.16)$$

for each of rows $l = 0,1,\dots,L-1$

2. Second, we compute a new rectangular array $G(l,q)$ defined as

$$G(l,q) = W_N^{lq}\, F(l,q) \qquad \begin{matrix} 0 \le l \le L-1 \\ 0 \le q \le M-1 \end{matrix} \qquad (3.5.17)$$

3. Finally, we compute the L-point DFTs

$$X(p,q) = \sum_{l=0}^{L-1} G(l,q)\, W_L^{lp} \qquad (3.5.18)$$

for each column $q = 0,1,\dots,M-1$, of the array $G(l,q)$.

On the surface it may appear that the computational procedure outlined above is more complex than the direct computation of the DFT. However, let us evaluate the computational complexity of (3.5.15). The first step involves the computation of L DFTs, each of M-points. Hence this step requires LM^2 complex multiplications and $L\ \ (M-1)$ complex additions. The second step requires L complex multiplications. Finally, the third step in the computation requires ML^2 Complex multiplications and $M\ \ (L-1)$ complex additions. Therefore, the computational complexity is

Complex multiplications: $N(M+L+1)$ (3.5.19)

Complex additions: $N(M+L-2)$

where $N = M$. Thus the number of multiplications has been reduce from N^2 to $N(M+L+1)$ and the number of additions will reduce from $N(N-1)$ to $N(M+L-2)$.

For example, suppose that N= 1000 and we select $L = 2$ and $M = 500$. Then, instead of having to perform 10^6 Complex multiplications via direct computation of the DFT, this approach leads to 503,000 complex multiplications. This represents a reduction by approximately a factor of 2.

When N is highly composite number, that is, N can be factored into a product of prime numbers of the form

$$N = r_1 r_2 \dots r_v \qquad (3.5.20)$$

then the decomposition above can be repeated $(v - 1)$ More times. This procedure results in smaller DFTs, which, in turn, leads to a more efficient computational algorithm.

In effect, the first segmentation of the sequence $x(n)$ into a rectangular array of M columns with L elements in each column resulted in DFTs of sizes L and M. Further decomposition of the data in effect involves the segmentation of each row (or column) into smaller rectangular arrays which result in smaller in DFTs. This procedure terminates when N is factored into its prime factors.

To illustrate this computational procedure, let us consider the computation of an $N = 15$ point DFT. Since $N = 5 \times 3 = 15$, we select $L = 5$ and $M = 3$. In other words, we store the 15-point sequence $x(n)$ column-wise as follows:

Row 1:	$x(0,0) = x(0)$	$x(0,1) = x(5)$	$x(0,2) = x(10)$
Row 2:	$x(1,0) = x(1)$	$x(1,1) = x(6)$	$x(1,2) = x(11)$
Row 3:	$x(2,0) = x(2)$	$x(2,1) = x(7)$	$x(2,2) = x(12)$
Row 4:	$x(3,0) = x(3)$	$x(3,1) = x(8)$	$x(3,2) = x(13)$
Row 5:	$x(4,0) = x(4)$	$x(4,1) = x(9)$	$x(4,2) = x(14)$

Now, we compute the three-point DFTs for each of the five rows. This leads to the following 5×3 array:

$F(0,0)$	$F(0,1)$	$F(0,2)$
$F(1,0)$	$F(1,1)$	$F(1,2)$
$F(2,0)$	$F(2,1)$	$F(2,2)$
$F(3,0)$	$F(3,1)$	$F(3,2)$
$F(4,0)$	$F(4,1)$	$F(4,2)$

Then next step is to multiply each of the terms $F(l, q)$ by the phase factors $W_N^{lq} = W_{15}^{lq}$, $0 \le l \le 4$ and $0 \le q \le 2$. This computation results in the 5×3 array:

Column 1	Column 2	Column 3
$G(0,0)$	$G(0,1)$	$G(0,2)$
$G(1,0)$	$G(1,1)$	$G(1,2)$
$G(2,0)$	$G(2,1)$	$G(2,2)$
$G(3,0)$	$G(3,1)$	$G(3,2)$
$G(4,0)$	$G(4,1)$	$G(4,2)$

The final step is to compute the five-point DFTs for each of the three columns.This computation yields the desired values of the DFT in the form

$x(0,0) = x(0)$	$x(0,1) = x(1)$	$x(0,2) = x(2)$
$x(1,0) = x(3)$	$x(1,1) = x(4)$	$x(1,2) = x(5)$
$x(2,0) = x(6)$	$x(2,1) = x(7)$	$x(2,2) = x(8)$
$x(3,0) = x(9)$	$x(3,1) = x(10)$	$x(3,2) = x(11)$
$x(4,0) = x(12)$	$x(4,1) = x(13)$	$x(4,2) = x(14)$

It is interesting to view the segmented data sequence and the resulting DFT in terms of one-dimensional arrays. When the input sequence x(n) and the output DFT X(k) in the two-dimensional arrays are read across from row 1 through row 5, we obtain the following sequences:

INPUT ARRAY

$$x(0)\ \ x(5)\ \ x(10)\ \ x(1)\ \ x(6)\ \ x(11)\ x(2)\ \ x(7)\ \ x(12)\ \ x(3)\ \ x(8)\ \ x(13)\ \ x(4)\ \ x(9)\ \ x(14)$$

OUTPUT ARRAY

$$x(0)\ \ x(1)\ \ x(2)\ \ x(3)\ \ x(4)\ \ x(5)\ \ x(6)\ \ x(7)\ \ x(8)\ \ x(9)\ \ x(10)\ \ x(11)\ \ x(12)\ \ x(13)\ \ x(14)$$

We observed that the input data sequence is shuffled from the normal order in the computation of the DFT. On the other hand, the output sequence occurs in normal order. In this case the rearrange of the input data array is due to the segmentation of the one-dimensional array into a rectangular array and the order in which the DFTs are computed. This shuffling of either the input data sequence or the output DFT sequence is a characteristic of most FFT algorithms.

To summarize, the algorithm that we have introduced involves the following computations:

Algorithm 1

1. Store the signal column-wise.
2. Compute the M-point DFT of each row.
3. Multiply the resulting array by the phase factors W_N^{l} .
4. Compute the L-point DFT of each column
5. Read the resulting array row-wise.

An additional algorithm with a similar computational structure can be obtained if the input signal is stored row-wise and the resulting transformation is column-wise. In this case we selected as $n = Ml + m \qquad k = qL + p$ (3.5.21)

This choice of indices leads to the formula for the DFT in the form

$$X(p,q) = \sum_{m=0}^{M-1}\sum_{l=0}^{L-1} x(l,m) W_N^{p} \quad W_L^{p} \quad W_M^{q} \quad = \sum_{m=0}^{M-1} W_M^{m} \left[\sum_{l=0}^{L-1} x(l,m)\, W_L^{l}\right] W_N^{m} \qquad (3.5.22)$$

Thus we obtain a second algorithm.

Algorithm 2

1. Store the signal row-wise.
2. Compute the L −point DFT at each row.
3. Multiply the resulting array by the factors W_N^{p}
4. Compute the M −point DFT of each row.
5. Read the resulting array column-wise.

The two algorithm given above have the same complexity. However, they differ in the arrangement of the computations. In the following sections we exploit the divide-and-conquer approach to drive fast algorithms when the size of the DFT is restricted to be a power of 2 or a power of 4.

3.5.3 Radix-2 FFT Algorithms

In the preceding section we described two algorithms for efficient computation of the DFT based on the divide-and-conquer approach. Such an approach is applicable when the number

N of data points is not a prime. In particular, the approach is very efficient when N is highly composite, that is, when N can be factored as $N = r_1 r_2 \dots r_\nu$, where the $\{r_j\}$ are prime.

Of particular importance as the case in which $r_1 = r_2 = \cdots = r_\nu \equiv r$, so that $N = r^\nu$. In such a case the DFTs are of size r, so that the computation of the N −point DFT has a regular pattern. The number r is called radix of the FFT algorithm.

Radix-2 algorithms are by far the most widely used FFT algorithm.

Let us consider the computation of the $N = 2^\nu$ point DFT by the divide-and-conquer approach specified by (6.1.16) through (6.1.18). We select $M = N/2$ And $L = 2$. This selection results in a split of the N −point data sequence into two …..-point data sequences $f_1(n)$ and $f_2(n)$ corresponding to the even-numbered and odd-numbered samples of $x(n)$, respectively, that is,

$$f_1(n) = x(2n); \qquad f_2(n) = x(2n+1); \qquad n = 0,1,\dots,\frac{N}{2}-1 \qquad (3.5.23)$$

Thus $f_1(n)$ and $f_2(n)$ are obtained by decimating $x(n)$ by a factor of 2, and hence the resulting FFT algorithm is called a decimation-in-time algorithm.

Now the N-point DFT can be expressed in terns of the DFTs of the decimated sequences as follows:

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)\, W_N^k \qquad k = 0,1,\dots,N-1 \\ &= \sum_{n\, e} x(n)\, W_N^k + \sum_{n\, o} x(n)\, W_N^k \qquad (3.5.24) \\ &= \sum_{m=0}^{(N/2)-1} x(2m)\, W_N^{2m} + \sum_{m=0}^{(N/2)-1} x(2m+1)\, W_N^{k(2m+1)} \end{aligned}$$

But $W_N^2 = W_{N/2}$ With this substitution, (3.5.24) can be expressed as

$$\begin{aligned} X(k) &= \sum_{m=0}^{(N/2)-1} f_1(m)\, W_{N/2}^k + W_N^k \sum_{m=0}^{(N/2)-1} f_2(m)\, W_{N/2}^k \\ &= F_1(k) + W_N^k F_2(k) \qquad 0,1,\dots,N-1 \qquad (3.5.25 \end{aligned}$$

where $F_1(k)$ and $F_2(k)$ are the $N/2$-point DFTs of the sequences $f_1(m)$ and $f_2(m)$, respectively.

Since $F_1(k)$ and $F_2(k)$ are periodic, with period $N/2$, we have $F_1(k+N/2) = F_1(k)$ and $F_2(k+N/2) = F_1(k)$. In addition, the factor $W_N^{k+N/2} = -W_N^k$. Hence (3.5.25) can be expressed as

$$X(k) = F_1(k) + W_N^k F_2(k) \qquad k = 0,1,\dots,\frac{N}{2}-1 \qquad (3.5.26)$$

$$X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k F_2(k) \qquad k = 0,1,\dots,\frac{N}{2}-1 \qquad (3.5.27)$$

We observed that the direct computation of $F_1(k)$ requires $(N/2)^2$ complex multiplications. The same applies to the computation $F_2(k)$ Furthermore, there are $N/2$ additional complex multiplications required to compute $W_N^k F_2(k)$ Hence the computation of $X(k)$ requires $2(N/2)^2 + N/2 = N^2/2 + N/2$ complex multiplications. The first step results in a reduction of the number of multiplications from N^2 To $N^2/2 + N/2$ which is about a factor of 2 for N large.

To be consistent with our previous notation, we may define

$$G_1(k) = F_1(k) \qquad k = 0,1,\dots,\frac{N}{2}-1$$

$$G_2(k) = W_N^k F_2(k) \qquad k = 0,1,\dots,\frac{N}{2}-1$$

Then the DFT $X(k)$ may be expressed as

$$X(k) = G_1(k) + G_2(k); \quad X\left(k+\frac{N}{2}\right) = G_1(k) - G_2(k) \quad k = 0,1,\dots,\frac{N}{2}-1 \qquad (3.5.28)$$

Having performed the decimation-in-time once, we can repeat the process for each of the sequences $f_1(n)$ and $f_2(n)$. Thus $f_1(n)$ would result in the two $N/2$- point sequences

$$v_1\ (n) = f_1(2n); \ \ v_1\ (n) = f_1(2n+1) \qquad n = 0,1,\dots,\frac{N}{4}-1 \qquad (3.5.29)$$

and $f_2(n)$ would yield

$$v_2\ (n) = f_2(2n); \ \ v_2\ (n) = f_2(2n+1) \qquad n = 0,1,\dots,\frac{N}{4}-1 \qquad (3.5.30)$$

By computing $N/4$ -point DFTs, we would obtain the $N/2$-point DFTs $F_1(k)$ and $F_2(k)$ from the relations

$$F_1(k) = V_1\ (k) + W_{N/2}^k V_1\ (n); \ \ F_1\left(k+\frac{N}{4}\right) = V_1\ (k) - W_{N/2}^k V_1\ (n) \quad k = 0,1,\dots,\frac{N}{4}-1$$

$$F_2(k) = V_2\ (k) + W_{N/2}^{k} V_2\ (n);\ \ F_2\left(k + \frac{N}{4}\right) = V_2\ (k) - W_{N/2}^{k} V_2\ (n)\quad k = 0,1,\dots,\frac{N}{4} - 1$$

where the $\{V_i\ (k)\}$ are the $N/4$ -point DFTs of the sequences $\{v_i\ (n)\}$

Table3.2 Computational complexity for direct computation of the DFT *vs.* FFT algorithm

Number of Points N	Complex Multiplications in Direct Computation N^2	Complex Multiplications in FFT Algorithm, $(N/2)\log_2 N$	Speed improvement Factor
4	16	4	4.0
8	64	12	5.3
16	256	32	8.0
32	1,024	80	12.8
64	4,096	192	21.3
128	16,384	448	36.6
256	65,536	1,024	64.0
512	262,144	2,304	113.8
1,0242	1,048,576	5,120	204.8

We observe that the computation of $\{v_i\ (k)\}$ requires $4(N/4)^2$ multiplications and hence the computation of $F_1(k)$ and $F_2(k)$ can be accomplished with $N^2/4 + N/2$ complex multiplications. An additional $N/2$ complex multiplications are required to compute $X(k)$ from $F_1(k)$ and $F_2(k)$. Consequently, the total number of multiplications is reduced approximately by a factor of 2 again to $N^2/4 + N$.

The decimation of the data sequence can be repeated again and again until the resulting sequences are reduced to one-point sequences. For $N = 2^v$, this decimation can be performed $v = \log_2 N$. Thus the total number of complex multiplications is reduced to $(N/2)\log_2 N$. The number of complex additions is $N\log_2 N$. Table 6.1 presents a comparison of the number of complex multiplications in the FFT and in the direct computation of the DFT.

3.6 Some Practical Guidelines

In general, the DFT is only an approximation to the actual (Fourier series or transform) spectrum of the underlying analog signal. The DFT spectral spacing and DFT magnitude is affected by the choice of sampling rate and how the sample values are chosen. The DFT phase is affected by the location of sampling instants. The DFT spectral spacing is affected by the sampling duration. Here are some practical guidelines on how to obtain samples of an analog signal $x(t)$ for spectrum analysis and interpret the DFT (or DFS) results.

Choice of sampling instants:

The defining relation for the DFT (or DFS) mandates that samples of $x(n)$ be chosen over the range $0 \le n \le N-1$ (through periodic extension, if necessary). Otherwise, the DFT (or DFS) phase will not match the expected phase.

Choice of samples:

If a sampling instant corresponds to a jump discontinuity, the sample value should be chosen as the midpoint of the discontinuity. The reason is that the Fourier series (or transform) converges to the midpoint of any discontinuity.

Choice of a frequency axis:

The computation of the DFT (or DFS) is independent of the sampling frequency S or sampling interval $t_s = 1/S$. However, if an analog signal is a sampled at a sampling rate S, its spectrum is periodic with period S. The DFT spectrum describes one period (N samples) of this spectrum starting at the origin. For sampled signals, it is useful to plot the DFT (or DFS) magnitude and phase against the analog frequency $f = kS/N\ Hz,\ \ k = 0,1,\ldots,N-1$ (with spacing S/N). For discrete-time signals, we can plot the DFT against the digital frequency $F = k/N,\ \ k = 0,1,\ldots,N-1$ (with spacing $1/N$).

Choice of frequency range:

To compare the DFT results with conventional two-sided spectra, it is to be remembered that by periodicity, a negative frequency $-f_C$ (at the index $-k_C$) in the two-sided spectrum, corresponds to the frequency $S - f_C$ (at the index $-k_C$) in the (one-sided) DFT spectrum.

Identifying the highest frequency:

The highest frequency in the DFT spectrum corresponds to the folding index $k = 0.5N$ and equal $f = 0.5S$ Hz for the sampled analog signals. This highest frequency is also called the **folding frequency**. For purpose of comparison, it is sufficient to plot the DFT spectra only over $0 \le F < 0.5N$ (or $0 \le F < 0.5$ for the discrete-time signals or $0 \le f < 0.5S$ Hz for sampled analog signals)

Plotted reordered spectra:

The DFT (or DFS) may also be plotted as two-sided spectra to reveal conjugate symmetry about the origin by creating its periodic extension. This is equivalent to creating a reordered spectrum by relocating the DFT samples at indices past the folding index $k = 0.5N$ to the left of the origin (because $X(-\mathrm{k}) = X\left(N-k\right)$).

CHAPTER-4
WAVELET TRANSFORM

Before throwing light about Wavelet transform we will introduce some concept essential for the understanding the topic. Though the title of the chapter is indicating that an in depth discussion on the topics will be available here, but our presentation will be at introductory level.

4.1 WINDOW FUNCTION

A desired of a signal can be removed from the main signal by multiplying the original signal by another function, which is zero outside the interval desired.

Let $\phi(t) \in L^2(\mathbb{R})$ be a real-valued window function. Then the product $f(t)\phi(t-b) = f_b(t)$ will contain the information of $f(t)$ near $t = b$. In particular, if¤$(t) =$ ¡$_{[-²,²)}(t)$, then

$$f_b(t) = \begin{cases} f(t), & t \in [b - \imath, b + \imath) \\ 0, & o\ he\imath \end{cases} \tag{4.1}$$

By changing the parameter b we can slide the window function along the time axis to analyze the local behavior of the function $f(t)$ in different intervals.

The two most important parameters for a window function are its center and width; the latter is usually twice the radius. It is clear that the center and the standard wide of the window function in Fig. 4.1 are 0 and 2$\imath$, respectively. For a general window function $\phi(t)$, we define its center t^* as

$$t^* = \frac{1}{\|\phi(t)\|^2} \int_{-\infty}^{\infty} t\|\phi(t)\|^2\, d \tag{4.2}$$

and the root-mean-square (RMS) radius Δ_ϕ as

$$\Delta_\phi = \frac{1}{\|\phi(t)\|^2} \left[\int_{-\infty}^{\infty} t\|\phi(t)\|^2 d \right]^{1/2} \tag{4.3}$$

For the particular window of Fig. 4.1, it is easy to verify that $t^* = 0$ and $\Delta_\phi = \imath / \sqrt{3}$. Therefore, the RMS width is smaller than the standard width by $1 / \sqrt{3}$.

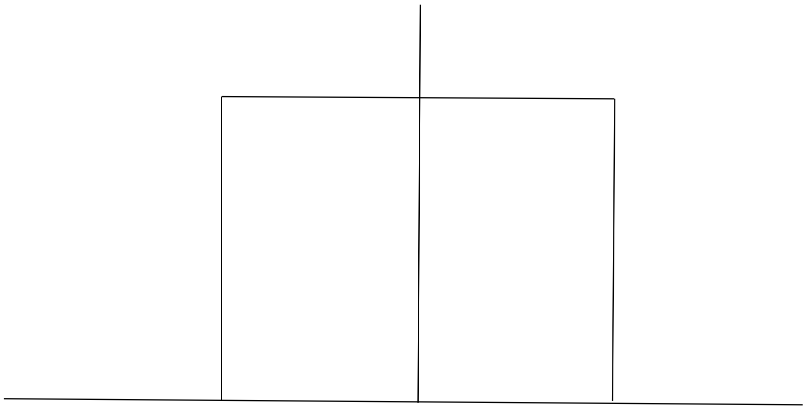

Figure 4.1- Characteristic Function

The function $\phi(t)$ describe above with finite Δ_{ϕ} is called a time window. Similarly, we can have a frequency window $\hat{\phi}(\omega)$ with center ω^* and the RMS radius $\Delta_{\hat{\phi}}$ define analogous to (4.2) and (4.3) as

$$\omega^* = \frac{1}{\|\hat{\phi}\|^2} \int_{-\infty}^{\infty} \omega |\hat{\phi}(\omega)|^2 \, d \qquad (4.4)$$

$$\Delta_{\phi} = \frac{1}{\|\hat{\phi}\|} \left[\int_{-\infty}^{\infty} (\omega - \omega^{*}) |\hat{\phi}(\omega)|^2 d \quad \right]^{1/2} \qquad (4.5)$$

As we know, theoretically a function cannot be limited in time and frequency simultaneously. However, we can have $\phi(t)$. such that both $\Delta_{\phi}$and $\Delta_{\hat{\phi}}$are both finite; in such a case the function $\phi(t)$ is called a time-frequency window. It is easy to verify that for the window of Fig. 4.1 $\omega^{*} = 0$ and $\Delta_{\hat{\phi}} = \infty$ This window is the best (ideal) time window but the worst (unacceptable) frequency window.

A figure of merit for the time-frequency window is its time-frequency-width product $\Delta_{\phi}\Delta_{\hat{\phi}}$, which is bounded from below by the uncertainty-principle and is given by

$$\Delta_{\phi}\Delta_{\hat{\phi}} \geq \frac{1}{2} \qquad (4.6)$$

Where the equality holds only when ¤ is of the Gaussain type.

4.2 DISCRETE SHORT-TIME FOURIER TRANSFORM

We indicate that we could obtain the approximate frequency contents of a signal $f(t)$ in the neighborhood of some desired location in time, say *t*=*b*, by first windowing the function using an appropriate window function $\phi(t)$ to produce the window function $f_b(t) = f(t)\phi(t-b)$ and then taking the Fourier transform of $f_b(t)$. This is the short-time Fourier transform (STFT). Formally, we can define the STFT of a function f(t) with respect to the window function $\phi(t)$ evaluated at the location (b, ξ). In the time-frequency plane as

$$G_\phi f(b_n, \xi_n) \approx h \sum_{k=0}^{N-1} f(t_k)\ \phi(t_k - b_n) e^{-j\xi_n t_k} \qquad (4.2.1)$$

where $$t_k = b_k = kh, \qquad k = 0,1, \dots, N-1 \qquad (4.2.2)$$

and $$\xi_n = \frac{2\pi}{Nh}, \qquad n = -\frac{N}{2}, \dots, \frac{N}{2} \qquad (4.2.3)$$

In particular, when *h*=1, we have

$$G_\phi f(n, \xi_n) \approx \sum_{k=0}^{N-1} f(k)\ \phi(k-n) e^{-j(2\pi \quad /N)} \qquad (4.2.4)$$

4.3 CONTINUOUS WAVELET TRANSFORM

The STFT one of many ways to generate a time frequency analysis of signals. Another linear transform that provides such analyses is the integral (or continuous) wavelet transform. The terms continuous wavelet transform (CWT) and integral wavelet transform (IWT) is normally used interchangeably. Fixed time-frequency resolution of the short-time Fourier transform (STFT) poses a serious constrain in many applications. In additions, developments on the discrete wavelet transform (DWT) and the wavelet series (WS) make the wavelet approach more suitable than the STFT for signal and image processing. To clarify our points, let us observe that the radii Δ_ϕ and $\Delta_{\ddot{\phi}}$ of the window function for STFT do not depend upon location in the $t-\omega$ plane. For instance, if we choose $\phi(t) = g_\alpha(t)$, once α is fixed, so are Δg_α and $\Delta \ddot{g}_\alpha$, regardless of the window location in the $t-\omega$ plane. Once the window function is chosen, the time-frequency resolution is fixed throughout the processing. To understand the implications of such a fixed resolution, let us consider the chirp signal, as shown in the following figure, in which the frequency of the signal increases with time.

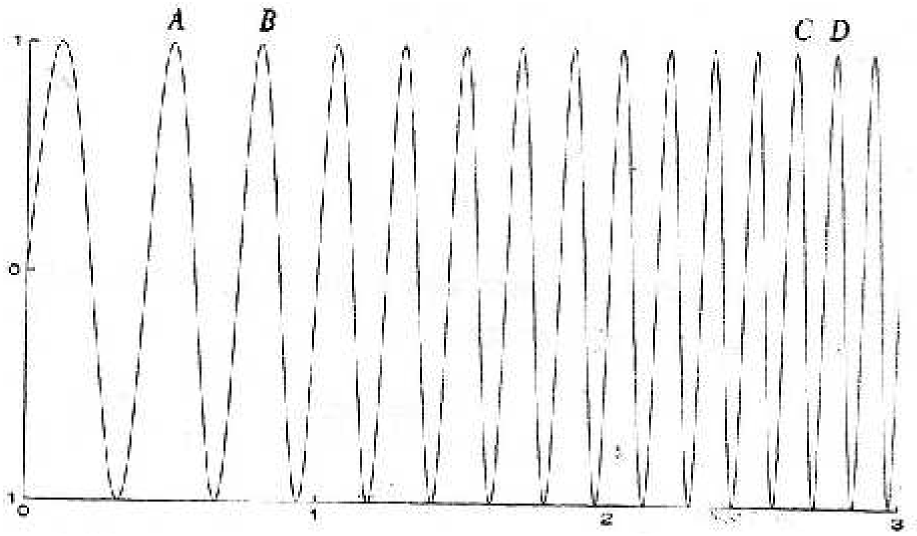

Figure 4.2- Chrip signal with frequency changing linearly with time

If we choose the parameters of the window function $\phi(t)$ [α in the case of $g_\alpha(t)$] such that Δ_ϕ In approximately equal to AB, the STFT as computed using (4.2.1) will be able to resolve the low-frequency portion of the signal better, while there will be poor resolution of the high-frequency portion. On the other hand, if $\Delta_\phi$is approximately equal to CD, the low frequency will not be resolve properly. Observe that if Δ_ϕ is very small, $\Delta_{\hat{\phi}}$ will be proportionally large, and hence the low-frequency part will be blurred.

Our objective is to devise a method that the can give good time-frequency resolution at an arbitrary location in the $t - \omega$ Plane. In other words, we must have a window function whose radius increases in time (reduces in frequency) while resolving the low-frequency contents, and the creases in time (increases in frequency) while resolving the high-frequency contents of a signal. This objective leads us to the development of wavelet functions $\psi(t)$.

4.3.1 Inverse Wavelet transform

Since the purpose of the inverse transform is to reconstruct the original signal/function from its transformed form, in the case of integral wavelet transform it involves a two-dimensional integration over the scale parameter *a* and the translation parameter *b*. The expression for the inverse wavelet transform is

$$f(t) = \frac{1}{C_{.}} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{1}{a^2} [W_{.} f(b,a)] \cdot {}_{b,a} da \qquad (4.3.1)$$

where $C_{\cdot}$ is a constant that depends on the choice of wavelet and is given by

$$C_{\cdot} = \int_{-\infty}^{\infty} \frac{\left|\hat{\cdot}(\mu)\right|^2}{|\mu|} d\mu < \infty \tag{4.3.2}$$

The condition (4.3.2), known as the *admissibility condition*, restricts the class of function that can be wavelets. In particular, It implies that all wavelets must have $\hat{\cdot}(0) = \int_{-\infty}^{\infty} \cdot\ (t)dt = 0$ in order to make the left hand side of (4.3.2) a finite number.

Equation (4.3.1) is essentially a superposition integral. Integration with respect to *a* sums all the combinations of the wavelet components at location *b*, while the integral with respect to *b* includes all locations along the *b*-axis. Since computation of the inverse wavelet transform is quite cumbersome and the inverse wavelet transform is used only for synthesizing the original signal, it is not used as f

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