Study on Distributive Lattice and Boolean Function

by

Joyanta Kumar Kundu

Roll No : 1451558

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Dedicated to My Parents

Jagobandhu Kundu & Nupur Kundu

and

Younger Sister Rupa Kundu.

Who have desired underprivileged life to continue my smile.

Declaration

This is to certify that the thesis work entitled "**Study on Distributive Lattice and Boolean Function**" has been carried out by Joyanta Kumar Kundu, in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part work of this work has not been submitted anywhere for the award of any degree or diploma.

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This is to certify that the thesis work submitted by Joyanta Kumar Kundu entitled "**Study on Distributive Lattice and Boolean Function**" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in November 2016.

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Abstract

In this thesis the nature of distributive lattice and Boolean function is studied. Distributive lattices have played many roles in the development of lattice theory. Distributive lattices have been studied by several authors including Abbott [6], Cornish [17], Nieminen [10].A poset $(L,\leq)$ is said to form a lattice if for every $a,b\in L$ if $\sup\{a,b\}$ and $\inf\{a,b\}$ exist in L. A lattice L is said to be distributive if $\forall a,b,c\in L$,

$$a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c) \text{ holds.}$$

In this thesis we give several results on distributive lattice and Boolean function which will certainly extend and generalize many results in lattice theory. We have proved that $N\times N$ is modular where N is the chain of naturals under usual $\leq$. We also generalize the following theorem of L.Nachbin[16], let L be a distributive lattice with 0 and 1 then L is Boolean iff all its prime ideals are unordered by set inclusion $\subseteq$.

Contents

CHAPTER 1

Preliminaries

1.1 Introduction:

In this chapter we have discussed the basic definition of relation. We recall some definitions and results on lattice, convex sublattice, complete lattice and complemented lattice. We consider this chapter as the base and background for the study of subsequent chapters.

1.2 Relations:

Definition (Relation): Let A and B be two non-empties set, any subset of $A \times B$ (Cartesian product) is called relation from A to B. The elements $a, b(a \in A, b \in B)$ are in relation with respect to R if $(a,b) \in R.$ For $(a,b) \in R,$ we will also write "aRb" or "$a \equiv b(R)$" and read as "a is related to b by R".

Example 1.2.1: Let $A = \{1,2,3\}$; $B = \{4,5\}$

Then $A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

$R_1 = \{(1,4),(1,5)\}$, $R_2 = \{(2,5)\}$, $R_3 = \{(3,4),(1,5)\}$ are all relations from A to B.

Definition (Inverse relation): Every relation R from A to B has an inverse relation R^{-1} from B to A which is defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed, i.e., permutated, belongs to R.

Example 1.2.2: Let $A = \{1,2,3\}$ and $B = \{a,b\}$.Then

$R = \{(1,a),(1,b),(3,a)\}$ is a relation from A to B. Then the inverse relation of R is

$$R^{-1} = \{(a,1),(b,1),(a,3)\}.$$

Definition (Reflexive relation): A relation R in a set A is called a reflexive relation if, for every $a \in A, (a,a) \in R$.

In other words, R is reflexive if every element in A is related to itself.

Example 1.2.3: Let $A = \{1,2,3\}$. Then

$$R = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$$

Here R is reflexive since $(1,1),(2,2)$ and $(3,3)$ belongs to the relation.

Definition (Symmetric relation): Let R be a subset of $A \times A$, i.e., let R be a relation in A. Then R is called a symmetric relation if $(a,b) \in R$ implies $(b,a) \in R$

that is, if a is related to b then b is also related to a.

Example 1.2.4: Let $A = \{1,2,3\}$. Then

$R = \{(1,1),(3,2),(2,3)\}$ Is symmetric relation.

Definition (Anti-symmetric relation): Let R be a subset of $A \times A$, i.e. let R be a relation in A. Then R is called a anti- symmetric relation if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$

In other words, if $a \neq b$ then possibly a is related to b or possibly b is related to a, but never both.

Remark: Let D denoted the diagonal line of $A \times A$, i.e., the set of all ordered pairs $(a,a) \in A \times A$.

Then a relation R in A is anti-symmetric if and only if

$$R \cap R^{-1} \subset D.$$

Example 1.2.5: Let $A = \{1,2,3\}$ .Then

$R_1 = \{(1,1)\}, R_2 = \{(1,2)\}$ both are anti-symmetric relation.

Definition (Transitive relation): A relation R in a set A is called a transitive relation if

$$(a,b) \in R \text{ and } (b,c) \in R \text{ implies } (a,c) \in R.$$

In other words, if a is related to b and b is related to c then a is related to c.

Example 1.2.6: Let $A=\{1,2,3\}$. Then

$R_1=\{(1,2),(2,2)\}, R_2=\{(1,2)\}$ both are transitive relation.

Definition (Equivalence relation): A relation R in a set A is an equivalence relation if

(1) R is reflexive , that is for every $a\in A, (a,a)\in R$

(2) R is symmetric, that is $(a,b)\in R$, ,implies $(b,a)\in R$

(3) R is transitive, that is $(a,b)\in R$ and $(b,c)\in R$ implies $(a,c)\in R$.

Example 1.2.7: Let $A=\{1,2,3\}$ be a set and $R=\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),$ $(2,3)\}$ be a relation of $A\times A$ then the relation is an equivalence relation, since

(1) R is reflexive, $(1,1),(2,2),(3,3)\in R$,

(2) R is symmetric, $(1,2),(2,1),(1,3),(3,1)\in R$ and

(3) R is transitive, $(2,1),(1,3),(2,3)\in R$.

1.3 Posets and Lattices:

Definition (Poset): A non-empty set P, together with a binary relation R is said to be a Partially Orderd set or a Poset if

(P1) aRa for every $a\in P$, i.e., R is reflexive.

(P2) aRb and bRa implies $a=b$, i.e., R is anti-symmetric, for $a,b\in P$

(P3) aRb and bRc implies aRc, i.e., R is transitive, for $a,b,c\in P$.

Remark: For our convenience, we use the symbol "$\leq$" in place of R. We read $\leq$ as "less than or equal to". Thus if P is a poset then we automatically assume that "$\leq$" is the partial ordered relation in P, unless other symbol is mentioned.

Examples 1.3.1: i) The set N of natural numbers under the usual $\leq$ is a poset.

ii) The set $X = \{2,4,8,16\}$ under the divisibility relation is a poset.

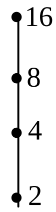


Fig- 1.1

Definition (Chain): If P is a poset in which every two members are comparable it is called a totally ordered set or a toset or a chain.

Thus if P is a chain and $x, y \in P$ then either $x \leq y$ or $y \leq x$.

Clearly also if x, y are distinct elements of a chain then either $x < y$ or $y < x$.

Definition (Greatest element): Let P be a poset. If $\exists$ an element $a \in P$ s.t. $x \leq a$ for all $x \in P$ then a is called greatest or unit element of P. Greatest element if exists, will be unique.

Definition (Least element): Let P be a poset. If $\exists$ an element $b \in P$ s.t. $b \leq x$ for all $x \in P$ then b is called least or zero element of P. Least element if exists, will be unique.

Example 1.3.2: Let $X = \{1,2,3\}$. Then $(P(X), \subseteq)$ is a poset.

Let $A = \{\phi, \{1,2\}, \{2\}, \{3\}\}$ then $(A, \subseteq)$ is a poset with φ as least element. A has no greatest element. Let $B = \{\{1,2\}, \{2\}, \{3\}, \{1,2,3\}\}$ then B greatest element $\{1,2,3\}$ but no least elements. If $C = \{\phi, \{1\}, \{2\}, \{1,2\}\}$ then C has both least and greatest elements namely, ϕ and $\{1,2\}$

Definition (Maximal element): An element a in a poset P is called maximal element of P if $a < x$ for no $x \in P$.

Example 1.3.3: In the poset $\{2,3,4,6,7,21\}$ under divisibility 4,6 and 21 are three maximal elements (none being the greatest).

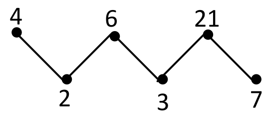


Fig- 1.2

Definition (Minimal element): An element b in a poset P is called a minimal element of P if $x < b$ for no $x \in P$.

Definition (Upper bound of a set): Let S be a non empty subset of a poset P. An element $a \in P$ is called an upper bound of S if $x \le a$ $\forall x \in S$.

Definition (Supremum): If a is an upper bound of S s.t. $a \le b$ for all upper bounds b of S then a is called least upper bound $(l.u.b)$ or supremum of S. We write $\sup S$ or supremum S.

It is clear that there can be more upper bound of a set. But $\sup$, if it exists, will be unique.

Definition (Lower bound): An element $a \in P$ will be called lower bound of S if $a \le x$, $\forall x \in S$.

Definition (Infimum): If a is a lower bound of a set S. Then a will be called greatest lower bound $(g.l.b)$ or Infimum S $(\inf S)$ if of a set $b \le a$ for all lower bounds b of S.

Example 1.3.4: Let $(Z, \le)$ be the poset of integers

Let $S = \{\cdots, -2, -1, 0, 1, 2)$ then $2 = \sup S$

Again the poset $(R, \le)$ of real numbers if $S = \{x \in R \mid x < 0, x \ne 0\}$ the $\sup S = 0$ (and it does not belong to S).

Definition (Lattice): A poset $(L,\leq)$ is said to form a lattice if for every $a,b\in L$, $\sup\{a,b\}$ and $\inf\{a,b\}$ exist in L.

In that case, we write

$\sup\{a,b\} = a\vee b$ (read *a* join *b*)

$\inf\{a,b\} = a\wedge b$ (read *a* meet *b*)

Other notations like $a+b$ and $a\cdot b$ or $a\cup b$ and $a\cap b$ are also used for $\sup\{a,b\}$ and $\inf\{a,b\}$.

Definition (Algebraic definition of a lattice): A set L together with two binary operation '$\wedge$' (meet) and '$\vee$' (join) is called a lattice if it satisfies the following identities

(i) idempotent law $\forall a\in L,\quad a\wedge a = a,\quad a\vee a = a$

(ii) commutative law $\forall a,b\in L,\quad a\wedge b = b\wedge a$ and $a\vee b = b\vee a$

(iii) associative law $\forall a,b,c\in L,\quad a\wedge(b\wedge c) = (a\wedge b)\wedge c$ and $a\vee(b\vee c) = (a\vee b)\vee c$

(iv) absorption law $\forall a,b\in L,\quad a\wedge(a\vee b) = a$ and $a\vee(a\wedge b) = a$.

Example 1.3.5: Let X be a non empty set, then the poset $(P(X),\subseteq)$ of all subset is a lattice.

Here for $A,B\in P(X)$

$A\wedge B = A\cap B$ and $A\vee B = A\cup B$

As particular case, when $X = \{1,2,3\}$

$P(X) = \{\phi,\{1\},\{2\},\{3\},\{1,2\}\{1,3\},\{2,3\},\{1,2,3\}\}$

It represented by the following figure

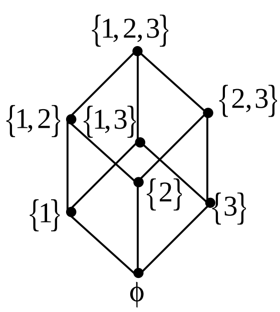


Fig- 1.3

Example 1.3.6: Every chain is a lattice. Since any two elements a, b of a chain are comparable, say $a \le b,$ we find

$a \wedge b = \inf\{a,b\} = a, \quad a \vee b = \sup\{a,b\} = b$.

Definition (Dual): Let R be a relation defined on a set P . Then converse of R (denoted by $\overline{R}$) is defined by $a\ \overline{R}\ b \Leftrightarrow b R a$, $a,b \in P$. Let (P,R) be a poset then $(\overline{P},\overline{R})$ is called dual of P, where $\overline{P} = P$ and $\overline{R}$ is the converse of R.

Remark: If a statement ϕ is true in all posets, then its dual is also true in all posets. As an example take for ϕ the statement: "If $\sup H$ exist it is unique", we get as its dual; "If $\inf H$ exists it is unique".

Theorem 1.3.7: Show that a poset is a lattice iff it is algebraically a lattice.

Proof: Clearly L is a non empty set.

So set $a \wedge b = \inf\{a,b\}$ and $a \vee b = \sup\{a,b\}$

Then $a \wedge a = \inf\{a,a\} = a$; $a \vee a = \sup\{a,a\} = a$

So $\wedge$ and $\vee$ are idempotent

$a \wedge b = \inf\{a,b\} = \inf\{b,a\} = b \wedge a$

$a \vee b = \sup\{a,b\} = \sup\{b,a\} = b \vee a$

$\because \wedge$ and $\vee$ are commutative.

Next, $a \wedge (b \wedge c) = \inf\{a, b \wedge c\} = \inf\{a, \inf\{b,c\}\}$

$= \inf\{\inf\{a,b\}, c\} = \inf\{a \wedge b, c\}$

$= (a \wedge b) \wedge c$

$a \vee (b \vee c) = \sup\{a, b \vee c\} = \sup\{a, \sup\{b,c\}\}$

$= \sup\{a, b \vee c\} = \sup\{a, \sup\{b,c\}\}$

$= \sup\{\sup\{a,b\}, c\} = \sup\{a \vee b, c\}$

$= (a \vee b) \vee c$

so $\wedge$ and $\vee$ are associative.

Finally, $a \wedge (a \vee b) = a \wedge \sup\{a,b\} = \inf\{a, \sup\{a,b\}\} = a$

$a \vee (a \wedge b) = a \vee \inf\{a,b\} = \sup\{a, \inf\{a,b\}\} = a$

Hence $\wedge$ and $\vee$ satisfy two Absorption identity

So $L = (L; ,\wedge, \vee)$ is a lattice.

Conversely Since $\wedge$ is idempotent i.e. $a \wedge a = a \quad \forall a \in L$

So $a \leq a$

$\therefore \leq$ is reflexive.

Since $\wedge$ is commutative

$\therefore a \wedge b = b \wedge a$

$\Rightarrow a = b \quad [\because a \wedge b = a \text{ and } a \vee b = b]$

So, $\leq$ is anti symmetric.

Let $a \leq b$ and $b \leq c$

Than $a = a \wedge b, \quad b = b \wedge c$

$= a \wedge (b \wedge c)$

$= (a \wedge b) \wedge c$

$= a \wedge c$

$\Rightarrow a = a \wedge c$

$\Rightarrow a \leq c$

So, $\leq$ is transitive

$\therefore (L, \leq)$ is a poset. ■

Theorem-1.3.8: Prove that a poset $(L, \leq)$ is a lattice if $\sup H$ and $\inf H$ exist for every non-void or non-empty subset H of L.

Proof : Let $(L, \leq)$ is a poset and H be a non void finite subset of L. If $H = \{a\}$, then $\sup H = \inf H = \{a\}$ follows from reflexivity of '$\leq$' and the definition of $\sup$ and $\inf$.

Let $H = \{a, b, c\}$ to show that $\sup H$ exist.

Let $d = \sup\{a, b\}$, $e = \sup\{d, c\}$

We claim that $e=\sup H$. First of all $a\le d, b\le d$ and $d\le e, c\le e$, therefore by transitivity $x\le e$. For all $x\in H$.

Secondly, if f is an upper bound of H, then $a\le f, b\le f$ and thus $d\le f$, also $c\le f$, so that $c,d\le f$. Therefore $e\le f$. Since $e=\sup\{d,c\}$

$\therefore e$ is the supremum of H .

If $H=\{a_0,a_1,a_2,\cdots,a_{n-1}\}$ then

$\sup\{\cdots\sup\{\sup\{a_0,a_1\}a_2\}\cdots a_{n-1}\}$ is the sup of H .

By duality, we conclude that $\inf H$ exist.

Hence a poset $(L,\le)$ is a lattice. ■

Theorem 1.3.9: A poset $(L,\le)$ is a lattice iff every non-empty finite subset of L has sup and inf .

Proof: Let $(L,\le)$ be a lattice and H be any non empty finite subset of L. Then there are several cases:

Case-1: If H has only one element, say a then $\inf H=\sup H=a$.

Case-2: If H has two elements, say a and b, then by definition lattice, $\sup H$ and $\inf H$ exist.

Case-3: Let H has three elements, say $H=\{a,b,c\}$, since by definition of a lattice, any two elements of L have sup and inf , let we take $d=\inf\{a,b\}$ and $e=\inf\{c,\inf\{a,b\}\}$ $=\inf\{c,d\}$

We must show, $e=\inf\{a,b,c\}$.

By definition of d and e, $d\le a,\ d\le b,\ e\le c,\ e\le d$

thus, $(e\le d,\ d\le a),\ (e\le d,\ d\le b),\ e\le c$

$\Rightarrow e\le a,\ e\le b,\ e\le c$ (transitivity)

$\Rightarrow e$ is any lower bound of $\{a,b,c\}$.

If f is any other lower bound of $\{a,b,c\}$, then $f\le a,\ f\le b,\ f\le c$

i.e. f is a lower bound of $\{a,b\}$ and $d=\inf\{a,b\}$ given $f\le d$

Again, $f \le c$ and $f \le d \Rightarrow f$ is a lower bound of $\{c, d\}$ and $e = \inf\{c, d\}$.

Given $f \le e$. Thus e is the greatest lower bound of $\{a, b, c\}$.

Hence $e = \inf\{a, b, c\} = \inf H$.

Similarly, $\sup H$ exists.

This result can similarly be extended to any finite number of elements in H. Indeed, if $H = \{a_1, a_2, a_3, \cdots, a_n\}$

Then $\inf H = \inf\{\inf\{\inf\{a_1, a_2\}, a_3\}, \cdots, a_n\}$

By duality, we can say, $\sup H$ exists.

Again let L be a non-empty set and H be any non-empty finite subset of L for which $\sup H$ and $\inf H$ exists.

We have to show that L is lattice.

Now, $\forall a, b \in L$, let we consider $H = \{a, b\}$.

By hypothesis, $\sup H = a \vee b$ and inf $\inf H = a \wedge b$ exist.

i.e. $a \vee b$ and $a \wedge b \in L$

Hence L is a lattice. ■

1.4 Convex sublattice and Complete lattice:

Definition (Bounded lattice): A Lattice with smallest and largest elements is called a bounded Lattice. Smallest element is denoted by '0' and largest element is denoted by 'l' or 'u'.

Example 1.4.1: The bounded subset of all real number under usual relation $\le$ is a bounded lattice.

Definition (Complete lattice): A lattice L is called a complete Lattice if every non-empty subset of L has its sup and inf in L.

Example 1.4.2: Set of all sub space of a vector space V is a complete Lattice under set inclusion.

Definition (Sublattice): Let $(L,\wedge,\vee)$ be a Lattice, A non empty subset S of L is called a sublattice of L if S itself is a lattice under same operation $\wedge$ and $\vee$ in L.

Example 1.4.3: Let $L=\{0,a,b,1\}$ be a lattice.

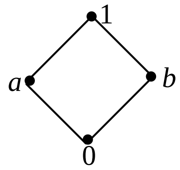


Fig- 1.4

Sublattice of L are: $\{0,a,b,1\},\{0\},\{a\},\{b\},\{1\},\{0,a\},\{0,b\},\{0,1\},\{a,1\},\{b,1\}$.

Definition (Convex sublattice): A sublattice S of a lattice L is called a convex sublattice of L. If for all $a,b\in S$, $[a\wedge b, a\vee b]\subseteq S$.

Example 1.4.4 : Let $L=\{0,a,b,c,1\}$ be a lattice.

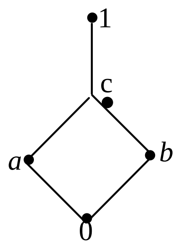


Fig- 1.5

Here $\{0,a,b,c\}$ is convex sublattice.

Remark : In the lattice $\{1,2,3,4,6,12\}$ under divisibility $\{1,6\}$ is a sublattice which is not convex as $2,3\in[1,6]$, but $2,3\notin\{1,6\}$ .Thus $[1,6]\not\subseteq\{1,6\}$.

Defination (Semilattice) : A poset is called a meet semilattice if for all $a, b \in P$,

$\inf\{a,b\}$ exist.

And a poset $(P, \leq)$ is called a join semilattice if for all $a, b \in P$, $\sup\{a,b\}$ exists.

Both the meet and join semilattice are called semilattice.

Definition (Algebraic definition of Semilattice) : A non-empty set P together with a binary composition $'\wedge'$ is called a meet semilattice and $'\vee'$ called a join semilattice, if for all $a, b, c \in P$.

(i) $a \wedge a = a, \; a \vee a = a$

(ii) $a \wedge b = b \wedge a, \; a \vee b = b \vee a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \; a \vee (b \vee c) = (a \vee b) \vee c$

Both meet and join semilattice are called semilattice.

Theorem-1.4.5 : Dual of a complete lattice is complete.

Proof : Let (L, ρ) be a complete lattice and let $(\overline{L}, \overline{\rho})$ be its dual. Then $(\overline{L}, \overline{\rho})$ is a lattice.

We have to show that $(\overline{L}, \overline{\rho})$ is complete lattice.

Let $\varphi \neq S \subseteq \overline{L}$ be any subset of $\overline{L}$.

Since L is complete, $\sup S$ and $\inf S$ exist in L.

Let, $a = \inf S$ in L.

Then $a \rho x, \forall x \in L$

$\Rightarrow x \overline{\rho} a, \forall x \in \overline{L}$

$\Rightarrow a$ is an upper bound of S in $\overline{L}$.

Let b be any other upper bound of S in $\overline{L}$

Then $x \overline{\rho} a, \forall x \in \overline{L}$

$\Rightarrow b \overline{\rho} x, \forall x \in L$

$\Rightarrow b \rho a$ as $a = \inf S$ in L.

$\Rightarrow a \overline{\rho} b$ or that 'a' is $l.u.b$ of S in $\overline{L}$

Similarly, we can show that $\sup S$ in L will be S in $\overline{L}$. Hence $(\overline{L}, \overline{\rho})$ is complete. ■

Theorem-1.4.6: A lattice is complete unless it has a subset which forms an infinite chain.

Proof: Suppose L is a lattice without infinite chain. Now we have to show that L is complete. For this purpose we need to show that, every subset of L has suprimum and infimum in L.

Let M be any subset of L and let

$M = \{x_0, x_1, x_2, \cdots\}$

If x_0 is an upper bound of M, then the theorem is proved. Now if x_0 is not an upper bound of M, then there is some $x_1 \in M$ such that $x_1 > x_0$.

Then $x_0 \vee x_1 \neq x_0$

Since $x_0 \vee x_1 \geq x_0$ so we deduce that $x_0 \vee x_1 > x_0$

$\Rightarrow x_0 < x_0 \vee x_1$

If $x_0 \vee x_1$ is an upper bound of M, then the theorem is proved. Now if $x_0 \vee x_1$ is not an upper bound of M, then there is some $x_2 \in M$

Such that $x_2 \neq x_0 \vee x_1$.

Then $x_0 \vee x_1 \vee x_2 \geq x_0 \vee x_1$.

Since $x_0 \vee x_1 \vee x_2 \neq x_0 \vee x_1$, so we deduce that

$x_0 \vee x_1 \vee x_2 > x_0 \vee x_1$

$\Rightarrow x_0 \vee x_1 < x_0 \vee x_1 \vee x_2$

If $x_0 \vee x_1 \vee x_2$ is an upper bound of M, then the theorem is proved.

If $x_0 \vee x_1 \vee x_2$ is not an upper bound of M, then there is an $x_3 \in M$

Such that $x_3 \neq x_0 \vee x_1 \vee x_2$. and such that

$$x_0 \vee x_1 \vee x_2 < x_0 \vee x_1 \vee x_2 \vee x_3$$

Proceeding in this way we get two cases:

Case-1 : The process continue on and on or, the lattice has an infinite ascending chain

$$x_0 < x_0 \vee x_1 < x_0 \vee x_1 \vee x_2 < x_0 \vee x_1 \vee x_2 \vee x_3 < \cdots$$

Case-2 : The process stops at certain stage Given that L has no infinite chain, so first cas can not occur. Therfore case-2 must occur.

Suppose the chain stops at $x_0 \vee x_1 \vee x_2 \vee \cdots \vee x_r$

Then $\overset{r}{\underset{i=0}{\vee}} x_i$ is an upper bound of M. Consider any other upper bound of M, say y.

$\therefore x_0 \le y, x_1 \le y, x_2 \le y, \cdots, x_r \le y$

$\therefore \overset{r}{\underset{i=0}{\vee}} x_i \le y$

So, $\overset{r}{\underset{i=0}{\vee}} x_i$ is the least upper bound of M, Hence M has supermum.

Similarly we can show that M has infimum. Hence a lattice without infinite chain is complete. ■

Theorem-1.4.7 : The set of all convex sublattice of a lattice under set inclusion is a sublattice.

Proof : Suppose L is a lattice.

Let us consider $C = \{A : [x, y] \subseteq A, \forall x, y \in A\}$

We have to show that $(C, \subseteq)$ is a lattice. $\forall P, Q \in C$

$P \wedge Q = P \bigcap Q$

$\forall x, y \in P \bigcap Q$

$x, y \in P$ and $x, y \in Q$

$\therefore [x, y] \subseteq P$ and $[x, y] \subseteq Q$

$\therefore [x, y] \subseteq P \bigcap Q$

So $P \wedge Q \in C$

Again suppose $P \vee Q = \{x : x \le p \vee q; p \in P, q \in Q\}$

Let $\alpha, \beta \in P \vee Q$, then $\exists$ $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ s.t.

$\alpha \le p_1 \vee q_1, \quad \beta \le p_2 \vee q_2$

Now $\forall \gamma \in [\alpha, \beta]$

So $\alpha \le \gamma \le \beta$

$\Rightarrow p_1 \vee q_1 \le \gamma \le p_2 \vee q_2$

$\Rightarrow \gamma \le p_2 \vee q_2$

$\therefore \gamma \in P \vee Q$

$\therefore [\alpha, \beta] \subseteq P \vee Q$

$\therefore P \vee Q \in C$

Hence C is a lattice.

i.e. Set of all convex sublattice of a lattice under set inclusion is a lattice. ■

Theorem-1.4.8 : Show that the poset $(P, \leq)$ is a lattice if it is a join and meet semilattice.

Proof: We know that a poset $(P, \leq)$ is called a lattice if it satisfied in the following axims:

(i) $\forall a, b \in P \Rightarrow a \wedge b \in P$ i.e. $\inf\{a, b\}$ exists in P.

(ii) $\forall a, b \in P \Rightarrow a \vee b \in P$ i.e. $\sup\{a, b\}$ exists in P.

Given that the poset $(P, \leq)$ be meet semilattice and join semilattice which implies

$\forall a, b \in P \Rightarrow \inf\{a, b\}$ exists in P {By the definition of meet semilattice}

and $\forall a, b \in P \Rightarrow \sup\{a, b\}$ exists in P {By the definition of join semilattice}

Since $\sup\{a, b\}$ and $\inf\{a, b\}$ exists in P.

So $(P, \leq)$ be a lattice. ■

Theorem 1.4.9 : A sub lattice S of a lattice L is a convex sublatice iff $\forall a, b \in S$ with $a \leq b$; $[a, b] \subseteq S$.

Proof : First suppose, S is a convex sublattice in L.

Then we have to show that $\forall a, b \in S$, $(a \leq b)$, $[a, b] \subseteq S$.

Let $\forall a, b \in S$ be any elements, then by definition of a convex sublattice, we have,

$$[a \wedge b, a \vee b] \subseteq S \text{ ....(1)}$$

But given that $a \leq b$

$\therefore a \wedge b = a, \quad a \vee b = b$

Therefore (i) becomes $[\mathrm{a}, \mathrm{b}] \subseteq \mathrm{S}$

Conversely suppose $\forall a, b \in S$ with $a \leq b$

$$[a,b] \subseteq S \;.....(2)$$

we have to show that S is convex sublattice in L.

Since S is a sublattice of L

So, by definition of a sublattice,

$$a \wedge b \in S \text{ and } a \vee b \in S, \quad \forall a,b \in S$$

Again, $\forall a,b$ we know.

$a \wedge b \leq a \vee b$

So by given condition. [i.e. (2) become]

$$[a \wedge b, a \vee b] \subseteq S$$

Therefore S is convex sublattice. ■

1.5 Complimented and Relatively complimented lattices :

Definition (Complimented lattice) : Let $[a,b]$ be an interval in a lattice L.

Let $x \in [a,b]$ be any element. If $\exists\ y \in L$ s.t. $x \wedge y = a, x \vee y = b$, we say y is a complement of x relative to $[a,b]$, or y is a relative complement of x in $[a,b]$.

Definition (Relatively complimented lattices) : If every element x of an interval $[a,b]$ has a least one complement relative to $[a,b]$, the interval $[a,b]$ is said to be complemented.

Further, if every interval in a lattice is complemented, the lattice is said to be relatively complemented.

Theorem 1.5.1 : Let A be a non-empty finite set. Show that $(\rho(A),\subseteq)$ is uniquely complemented lattice.

Proof : Let $A = \Phi$ finite set and $\rho(A)$ be the power set of A. We know $(\rho(A),\subseteq)$ form a lattice with least element Φ and greatest element A.

Any $X,Y \in \rho(A)$, $X \wedge Y = X \cap Y$ and $X \vee Y = X \cup Y$

Since $A \wedge (A-X) = A \cap (A-X) = \Phi$

$A \vee (A-X) = A \cup (A-X) = A$

We see $A-X$ is complemented of X relative to $[\Phi, \mathrm{A}]$

Then $\rho(A)$ is any complemented lattice. Suppose Y is any complemented of X then

$X \wedge Y = X \cap Y = \Phi$

$X \vee Y = X \cup Y = A$

ie, $X \cap Y = A \cap (A-X)$

$X \cup Y = A \cup (A-X)$

$Y = A - X$ ....(i)

or that $A-X$ is uniquely complemented of X .

So $(\rho(A), \subseteq)$ is an uniquely complemented lattice.

Now we prove $\rho(A)$ is also relative complemented.

Consider any interval $[X, Y]$ in $\rho(A)$

Let $Z \in [X, Y]$ be any number, Then

$Z \cap (X \cup (Y-Z)) = (Z \cap X) \cup (Z \cap (Y-Z)) = X \cup \Phi = X$

$Z \cup (X \cup (Y-Z)) = (Z \cup X) \cup (Y-Z) = Z \cup (Y-Z) = Y$

Showing that $X \cup (Y-Z)$ is the complemented of Z relative to $[X, Y]$.

Z is any element of any interval of $\rho(A)$.

Hence $\rho(A)$ is relative complemented. ■

Theorem 1.5.2 : Two bounded lattice A and B are complemented if and only if $A \times B$ is complemented.

Proof: Let A and B be complemented and suppose 0, 1 and $0', 1'$ are universal boundes of A and B respectively.

Then $(0, 0')$ and $(1, 1')$ will be least and greatest elements of $A \times B$

Let $(a, b) \in A \times B$ be any element.

Then $a, \in A, b \in B$ and as A, B are complemented, $\exists$ $a' \in A, b' \in B$ s.t.,

$a \wedge a' = 0,\ a \vee a' = 1,\ b \wedge b' = 0',\ b \vee b' = 1'$

Now $\quad (a,b)\wedge(a',b')=(a\wedge a',b\wedge b')=(0,0')$

$$(a,b)\vee(a',b')=(a\vee a',b\vee b')=(1,1')$$

Shows that (a',b') is complement of (a,b) in $A\times B$.

Hence $A\times B$ is complemented.

Conversely, let $A\times B$ be complemented.

Let $a\in A, b\in B$ be any elements.

Then $(a,b)\in A\times B$ and thus has a complement, say (a',b')

Then $(a,b)\wedge(a',b')=(0,0'),\quad (a,b)\vee(a',b')=(1,1')$

$\Rightarrow (a\wedge a',b\wedge b')=(0,0'),\quad (a\vee a',b\vee b')=(1,1')$

$\Rightarrow a\wedge a'=0,\ a\vee a'=1$

$b\wedge b'=0',\ b\vee b'=1'$

i.e., a' and b' are complements a and b respectively. Hence A and B are complemented. ■

Theorem 1.5.3 : Two lattice A and B are relatively complemented if and only if $A\times B$ is relatively complemented.

Proof: Let A,B be relatively complemented.

Let $[(a_1,b_1),\ (a_2,b_2)]$ be any interval of $A\times B$ and suppose (x,y) is any element of this interval.

Then $(a_1,b_1)\le(x,y)\le(a_2,b_2)\qquad a_1,a_2,x\in A\ \ b_1,b_2,y\in B$

$\Rightarrow a_1\le x\le a_2\ \ b_1\le y\le b_2$

$\Rightarrow x\in[a_1,a_2]$ an interval in A, $y\in[b_1,b_2]$ an interval in B.

Since A,B are relatively complemented x,y have complements relative to $[a_1,a_2]$ and $[b_1,b_2]$ respectively.

Let x' and y' be these complements. Then

$$x\wedge x'=a_1\quad y\wedge y'=b_1$$

$$x\vee x'=a_2\quad y\vee y'=b_2$$

Now $\quad (x,y)\wedge(x',y')=(x\wedge x',y\wedge y')=(a_1,b_1)$

$(x,y)\vee(x',y')=(x\vee x',y\vee y')=(a_2,b_2)$

$\Rightarrow (x',y')$ is complement of (x,y) related to $[(a_1,b_1),(a_2,b_2)]$

Thus any interval in $A\times B$ is complemented.

Hence $A\times B$ is relative complemented.

Conversely, let $A\times B$ be relatively complemented.

Let $[a_1,a_2]$ and $[b_1,b_2]$ be any intervals in A and B.

Let $x\in[a_1,a_2],\ y\in[b_1,b_2]$ be any elements.

Then $a_1\le x\le a_2,\ \ b_1\le y\le b_2$

$\Rightarrow (a_1,b_1)\le(x,y)\le(a_2,b_2)$

$\Rightarrow (x,y)\in[(a_1,b_1),(a_2,b_2)]$ an interval in $A\times B$

$\Rightarrow (x,y)$ has a complement, say (x',y') relative to this interval

Thus $\quad (x,y)\wedge(x',y')=(a_1,b_1)$

$(x,y)\vee(x',y')=(a_2,b_2)$

$\Rightarrow \quad (x\wedge x',y\wedge y')=(a_1,b_1)$

$(x\vee x',y\vee y')=(a_2,b_2)$

$\Rightarrow \quad x\wedge x'=a_1,\ \ x\vee x'=a_2$

$y\wedge y'=b_1,\ \ y\vee y'=b_2$

$\Rightarrow x'$ is complement of x relative to $[a_1,a_2]$

y' is complement of y relative $[b_1,b_2]$

Hence A,B are relatively complemented. ■

Theorem 1.5.4 : Dual of a complemented lattice is complemented.

Proof: Let (L,ρ) be a complemented lattice with $0,1$ as least and greatest elements. Let $(\overline{L},\overline{\rho})$ be the dual of (L,ρ). Then $1,0$ are least and greatest elements of $\overline{L}$.

Let $a\in\overline{L}=L$ be any element

Since $a\in L,\ L$ is complemented, $\exists\, a'\in L$ s.t.,

$$a \wedge a' = o, \quad a \wedge a' = 1 \text{ in } L$$

i.e., $0 = \inf\{a, a'\}$ in L

$\Rightarrow 0 \rho a, \; 0 \rho a'$

$\Rightarrow a \overline{\rho} 0, \;\; a' \overline{\rho} 0$ in $\overline{L}$

$\Rightarrow$ 0 is an upper bound of $\{a, a'\}$ in $\overline{L}$

If k is any upper bound of $\{a, a'\}$ in $\overline{L}$ then $a \overline{\rho} k, \;\; a' \overline{\rho} k$

$\Rightarrow k \rho a, k \rho a' \Rightarrow k \rho 0$ as 0 is inf .

$\Rightarrow 0 \overline{\rho} k$

i.e., 0 is $l.u.b$ $\{a, a'\}$ in $\overline{L}$

i.e., $a \vee a' = 0$ in $\overline{L}$

Similarly, $a \wedge a' = 1$ in $\overline{L}$

or that a' is complement of a in $\overline{L}$

Hence $\overline{L}$ is complemented. ∎

CHAPTER 2

Prime Ideal and Homomorphisms

2.1 Introduction :

In this chapter we discuss ideals, homomorphism, embedding mapping and kernel. We have proved the following theorem, lf L_1, L_2, M_1, M_2 are lattices such that $L_1 \cong M_1$ and $L_2 \cong M_2$ then

$$L_1 \times L_2 \cong M_1 \times M_2 \cong M_2 \times M_1 .$$

We also proved, if $\psi : L \to M$ is an onto homomorphism, where L, M are lattices and $0'$ is least element of M, then $\ker\psi$ is an ideal of L.

2.2 Basic Concept of Ideals :

Definition (Ideal) : A non-empty subset I of a lattice L is called an ideal of L if

(i) $\forall a, b \in I \Rightarrow a \vee b \in I$

and (ii) $\forall a \in I, \forall l \in L \Rightarrow a \wedge l \in I$ hold.

Note: If L is bounded then $\{0\}$ is always an ideal of L and it is called the zero ideal of L .s

Example 2.2.1: Let $L = \{1, 2, 5, 10\}$ be the lattice of factors of 10 under divisibility. Then $\{1\}, \{1, 2\}, \{1, 5\}, \{1, 2, 5, 10\}$ are all the ideal of L. But $\{5,10\}$ is not an ideal of L.

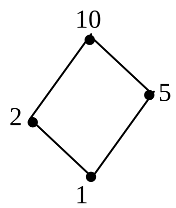


Fig- 2.1

Definition (Proper Ideal): An ideal I of the lattice is said to be proper ideal if $I \neq L$.

Example 2.2.2: Let $L = \{0, a, b, 1\}$ be a lattice whose Hasse diagram is given adjacent

Hence $I_1 = \{0\}, I_2 = \{0, a\}, I_3 = \{0, b\}$ each of them are proper ideal of L.

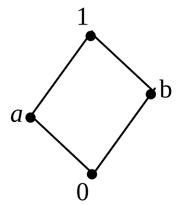


Fig- 2.2

Definition (Dual Ideal): A non empty subset I of a lattice L is called dual ideal (or filter) of L if

(i) $\forall a, b \in I \Rightarrow a \wedge b \in I$

(ii) $\forall a \in I, \forall\, l \in L \Rightarrow a \vee l \in I$.

Example 2.2.3 : Dual ideals are $\{1\}$, $\{1, c\}$, $\{1, c, a\}$, $\{1, c, b\}$, $\{1, c, a, b, 0\}$ in $L = \{0, a, b, c, 1\}$.

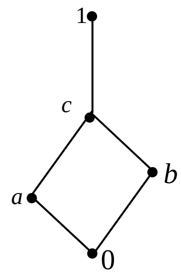


Fig- 2.3

Note: Dual ideal generated by a subset H of L is denoted by $[H]$.

Definition (Principal Ideal): Let L be a lattice and $a \in L$ be any element. The set $(a] = \{x \in L : x \leq a\}$ forms an ideal of L, is called principal ideal generated by a.

Example 2.2.4:

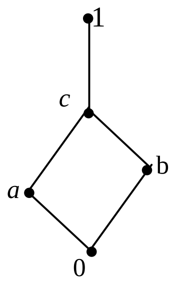


Fig- 2.4

In the figure, the principal ideal are

$$(0]=\{0\}$$
$$(a]=\{0,a\}$$
$$(b]=\{0,b\}$$
$$(c]=\{0,a,b,c\}$$
$$(1]=L$$

In a finite lattice, every ideal is a principal ideal.

Definition (Principal Dual Ideal) : Let L be a lattice and $a\in L$ be any element the set $[a)=\{x\in L: a\leq x\}$ forms a dual ideal of L is called the principal dual ideal generated by a.

Example 2.2.5 : In $L=\{1,2,5,10\}$ then $\{5,10\}$ is a principal dual ideal of L generated by 5.

Theorem 2.2.6 : A non-empty subset I of a lattice L is an ideal iff

(i) $a,b\in I \Rightarrow a\vee b\in I$

(ii) $a\in I$, $x\leq a\Rightarrow x\in I$.

Proof: Let I be an ideal of a lattice L. Then by definition of an ideal, condition (i) is satisfied.

Now let, $a\in I$, $x\leq a\Rightarrow x=a\wedge x.$ Then by definition of an ideal we get, $a\wedge x\in I$ i.e., $x\in I$. $[x\in L\,\text{since}\, I\subset L]$

i.e., condition (ii) is satisfied.

Here we show that.

$$a\in I,\, l\in L\Rightarrow a\wedge l\in I$$

Since $a \wedge l \leq a$ and $a \in I$, so from condition (ii) we get $a \wedge l \in I$. Hence I is an ideal. ■

2.3 Prime Ideal & Ideal Lattice:

Definition (Prime Ideal): An ideal I of a lattice L is called a prime ideal of L if I is properly contained in L and wherever $a \wedge b \in I$ then $a \in I$ or $b \in I$.

Example2.3.1: In the lattice $\{1,2,5,10\}$ under divisibility $\{1\}$ is not a prime ideal as $2 \wedge 5 = 1 \in \{1\}$, but $2,5 \notin \{1\}$. Here $\{1,2\}$ is a prime ideal.

Definition (Dual Prime Ideal): A proper dual ideal I of a lattice L is called a dual prime ideal if $a \vee b \in I \Rightarrow a \in I$ or $b \in I$.

Example 2.3.2 : In the lattice $L = \{1,2,5,10\}, \{5,10\}$ is a dual prime ideal of L.

Definition (Ideal lattice): The set of all ideals of a lattice L is called ideal lattice of L. It is denoted by $I(L)$.

Theorem 2.3.3: An ideal is a sublattice. Converse is not true.

Proof: Let I be an ideal of a lattice L. Also let $a, b \in I$

Then, by definition, $a \vee b \in I$. Again by definition

$$a \in I, b \in I \subseteq L \Rightarrow a \wedge b \in I$$

Hence, I is a sublattice of the lattice L.

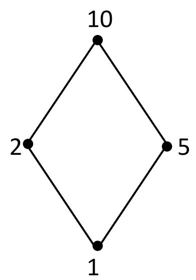


Fig: 2.5

We given here a counter example, let $L_{10} = \{1, 2, 5, 10\}$ be the lattice of factors of 10 under divisibility. Again let $I = \{5,10\}, I \subset L$. Moreover, I is a sublattice since, $5 \wedge 10 = 5 \in I$ and $5 \vee 10 = 10 \in I$. But I is not ideal of L_{10}, since $5 \in I$ and $2 \in L_{10}$ and $2 \wedge 5 = 1 \notin I$. ■

Theorem 2.3.4: Let I be a prime ideal of lattice L. iff $L - I$ is a dual prime ideal.

Proof: Since I is a prime ideal of L, so I is non-empty. Then $L - I$ is also non-empty proper subset of L.

Let $a, b \in L - I$, then $a, b \in L$

$a, b \notin I \Rightarrow a \wedge b \in L, a \wedge b \notin I$

$\Rightarrow a \wedge b \in L - I$

Again, Let $a \in L - I$ and $l \in L$

we need to show that $a \vee l \in L - I$

Now, $a \in L - I$ and $l \in L$

$\Rightarrow a \in L, \; a \notin I$ and $l \in L$

$\Rightarrow a \vee l \in L, \; a \notin I$

$\Rightarrow a \vee l \in L, a \vee l \notin I$

$\therefore a \vee l \in L - I$

$\Rightarrow L - I$ is a dual ideal.

We have to show that $L - I$ is a dual prime ideal.

Let, $a \vee b \in L - I$, then

$a \vee b \in L, a \vee b \notin I$

$\Rightarrow a, b \in L, \; a \notin I$ or $b \notin I$

$\Rightarrow a \in L - I$ or $b \in L - I$

$\Rightarrow L - I$ is a dual prime ideal.

Conversely, suppose $L - I$ is a dual prime ideal, So, $L - I$ is proper ideal and non-empty then I is also proper subset of L.

Let $a, b \in I \Rightarrow a, b \notin L - I$

$\Rightarrow a \vee b \notin L - I \quad [\because L - I$ is prime duel ideal]

$\Rightarrow a \vee b \in I$

Again, let $a \in I$ and $l \in L$

We need to show that $a \wedge l = L$

Now $a \in I$ and $l \in L$

$\Rightarrow a \notin L - I$ and $l \in L$

$\Rightarrow a \wedge l \notin L - I$

$\Rightarrow a \wedge l \in I$

Thus I is a ideal.

Again let $a \wedge b \in I$

$\Rightarrow a \wedge b \notin L - I$

$\Rightarrow a \notin L - I$ or $b \notin L - I$

$\Rightarrow a \in I$ or $b \in I$

$\therefore I$ is a prime ideal. ■

2.4 Homomorphism and Isomorphism:

Definition (Meet homomorphism): Let $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ be any two lattices then the map $\psi : L_1 \rightarrow L_2$ is called a meet homomorphism if $\forall a, b \in L$ $\psi : L_1 \rightarrow L_2$

$$\psi(a \wedge b) = \psi(a) \wedge \psi(b) \text{ holds.}$$

Example 2.4.1: Let $L_1 = \{0, a, b, 1\}$ and $L_2 = \{0, c, d, 1\}$ be two lattices. Let us define the map $\psi : L_1 \rightarrow L_2$ by the following way

Here $\psi(a \wedge b) = \psi(0) = 0, \quad a, b \in L_1$

$$= 0 \wedge c$$

$$= \psi(a) \wedge \psi(b)$$

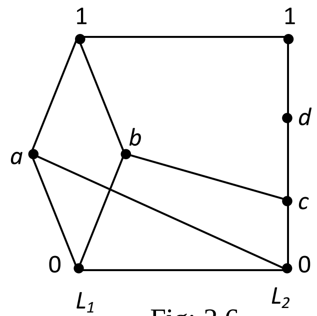


Fig: 2.6

Therefore, the mapping ψ is a meet-homomorphism.

Definition (Join homomorphism): Let $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ be any two lattices. Then the map $\psi : L_1 \rightarrow L_2$ is called a join-homomorphism if $\forall a, b \in L$,

$$\psi(a \vee b) = \psi(a) \vee \psi(b) \text{ holds.}$$

Example 2.4.2: Let us consider the map $\psi : L_1 \to L_2$ where $L_1 = \{0, a, b, 1\}$ and $L_2 = \{0, c, 1\}$ be two lattices and ψ be defined as follows.

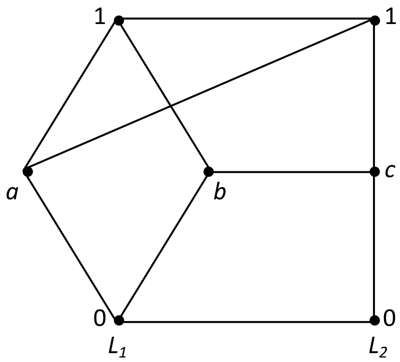


Fig: 2.6

Here. $\psi(a \vee b) = \psi(1), \quad a, b \in L_1$

$$= 1$$
$$= 1 \vee c$$
$$= \psi(\mathrm{a}) \vee \psi(\mathrm{b})$$

Therefore, the map ψ is a join-homomorphism.

Definition (Homomorphism): The map $\psi :\ (L_1, \wedge, \vee) \to (L_2, \wedge, \vee)$ is said to be homomorphism if $\forall a, b \in L_1, \psi(a \wedge b) = \psi(a) \wedge \psi(b)$ and $\psi(a \vee b) = \psi(a) \vee \psi(b)$ hold simultaneously.

Example 2.4.3: Let us consider the map $\psi : L_1 \to L_2$ where, $L_1 = \{0, a, b, 1\}$ and $L_2 = \{0, c, d, e, 1\}$ be two lattices and ψ be defined as follows.

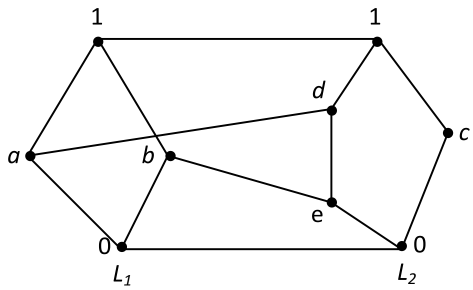


Fig: 2.7

Here, $\forall x, y \in L_1, \psi(x \wedge y) = \psi(x) \wedge \psi(y)$

and $\psi(x \vee y) = \psi(x) \vee \psi(y)$

hold simultaneously.

Hence the map ψ is a homomorphism.

Definition (Isomorphism): Let $(L_1; \wedge, \vee)$ and $(L_2; \wedge, \vee)$ be two lattices.

Then the map $\psi : L_1 \to L_2$ is called an isomorphism if

(i) ψ is a homomorphism.

and(ii) ψ is one-one and onto.

Note: (i) If ψ is an isomorphism from L to L we call it an automorphism.

(ii) A homomorphism from L to L is called endomorphism If $\psi : L \to L$ is onto homomorphism.

(iii) If the map $\psi : L_1 \to L_2$ is an homomorphism, then L_2 is said to be the homomorphic image of L_1.

Theorem 2.4.4: If L_1, L_2, M_1, M_2 are lattices such that $L_1 \cong M_1$ and $L_2 \cong M_2$ then show that $L_1 \times L_2 \cong M_1 \times M_2 \cong M_2 \times M_1$.

Proof: Let $f : L_1 \to M_1$ and $g : L_2 \to M_2$ be the given isomorphism.

Define $\psi : L_1 \times L_2 \to M_1 \times M_2$, s.t.,

$\psi((a,b) = (f(a), g(b))$

Then $\psi((a,b)) = \psi((c,d))$

$\Leftrightarrow$ $(f(a), g(b)) = (f(c), g(d))$

$\Leftrightarrow$ $f(a) = f(c),\ g(b) = g(d)$

$\Leftrightarrow$ $a = c, b = d$

$\Leftrightarrow$ $(a,b) = (c,d)$

Shows that ψ is well defined 1-1 map.

Again, $\psi((a,b) \wedge (c,d)) = \psi((a \wedge c, b \wedge d))$

$= (f(a \wedge c), g(b \wedge d))$

$= (f(a) \wedge f(c), g(b) \wedge g(d))$

$= (f(a), g(b)) \wedge ((f(c), g(d))$

$= \psi((a,b)) \wedge \psi((c,d))$

Similarly, $\psi((a,b) \vee (c,d)) = \psi((a,b)) \vee \psi((c,d))$

showing thereby that ψ is a homomorphism.

Finally, for any $(m_1, m_2) \in M_1 \times M_2$, since $m_1 \in M_1$ & $m_2 \in M_2$ and f, g are onto,

$\exists\ l_1 \in L_1$, $l_2 \in L_2$ s.t., $f(l_1) = m_1$, $g(l_2) = m_2$

and $\psi((l_1, l_2)) = (f(l_1), g(l_2)) = (m_1, m_2)$

shows that ψ is onto and hence an isomorphism.

To show $M_1 \times M_2 \cong M_2 \times M_1$, we can define

$\varphi : M_1 \times M_2 \to M_2 \times M_1$ s.t.,

$\varphi : ((\mathrm{m}_1, \mathrm{m}_2)) = (\mathrm{m}_2, \mathrm{m}_1)$

It is now easy to verify that φ is an isomorphism. ■

Theorem 2.4.5: I is a prime ideal of L iff there is a homomorphism ψ of onto C_2 with $I = \psi^{-1}(0)$.

Proof: Let I be a prime ideal and define ψ by

$\psi(x) = 0$ if $x \in I$

and $\psi(x) = 1$ if $x \notin I$

If $a,b \in I$ then $a \vee b \in I$ and I is an ideal.

No, $a,b \in I \Rightarrow \psi(a) = 0$ and $\psi(b) = 0$

Also $a \vee b \in I \Rightarrow \psi = \psi(a) \vee \psi(b)$

Therefore, $\psi(a \vee b) = \psi(a) \vee \psi(b)$

Again let $a,b \notin I$ then since I is prime ideal.

So, $a \wedge b \notin I \Rightarrow \psi(a) = \psi(b) = 1$ and $\psi(a \wedge b) = 1$

Consequently, $\psi(a \wedge b) = 1 = \psi(a) \wedge \psi(b)$

So ψ is homomorphism.

Conversely, let ψ be a homomorphism of L onto C_2 and $I = \psi^{-1}(0)$.

If $a,b \notin I$, then

$\psi(a) = \psi(b) = 1$

Thus $\psi(a \wedge b) = \psi(a) \wedge \psi(b) = 1$

Therefore, $a \wedge b \notin I$

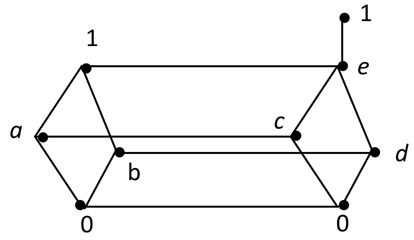
Hence I is a prime ideal. ■

2.5 Embedding mapping and Kernel :

Definition (Embedding mapping): Let L_1, L_2 be Lattices where $L_1 \subseteq L_2$. A one-one homomorphism $\psi : L_1 \to L_2$ is called an imbedding or embedding mapping.

In that case, it is said that L_1 is embedded in L_2.

Example 2.5.1: Let us consider the map $\psi : L_1 \to L_2$ where $L_1 \subseteq L_2$, $L_1 = \{0, a, b, 1\}$ and $L_2 = \{0, c, d, e, 1\}$ be two lattice and ψ be defied as ψ is one one and L_1 embedded in L_2.

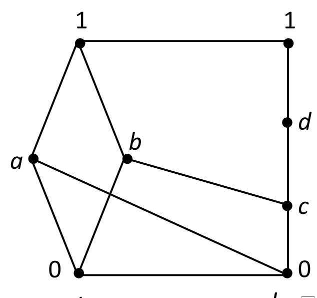


L_1 Fig: 2.9 L_2

Definition (Kernel of ψ): Let the map $\psi :\ L_1 \to L_2$ be onto homomorphism and $0'$ be the least element of L_2. Then the set $\{x \in L_1 : \psi(x) = 0'\}$ is said to be the kernel of ψ and is denoted by $\ker \psi$.

Note: If L_2 does not have the zero or least element, then ker $\ker \psi$ does not exist.

Example 2.5.2: Let $L_1 = \{0, a, b, 1\}, L_2 = \{0', c, d, 1\}$ and a map $\psi : L_1 \to L_2$ defined as then $\ker \psi = \{0, a\}$



L_1 L_2

Fig: 2.10

Theorem 2.5.3: If $\psi : L \to M$ is an onto homomorphism, where L, M are lattices and $0'$ is least element of M, then $\ker\psi$ is an ideal of L.

Proof: Since ψ is onto, $0' \in M$, thus $\ker\psi \neq \varphi$ as pre mage $0'$ exists in L.

Now $x, y \in \ker\psi \Rightarrow \psi(x) = 0' = \psi(y)$

$\psi(x \vee y) = \psi(x) \vee \psi(y) = 0' \vee 0' = 0' \Rightarrow x \vee y \in \ker\psi$

Again $x \in \ker\psi$, $l \in L$, gives $\psi(x) = 0'$

Also $\psi(x \wedge l) = \psi(x) \wedge \psi(y) = 0' \wedge l = 0'$

$\Rightarrow$ $x \wedge l \in \ker\psi$

Hence $\ker\psi$ is an ideal of L. ■

Theorem 2.5.4: Let L be a lattice, M be a finite chain and $L \to M$ is an onto homomorphism. Show that $\ker\psi$ is prime ideal of L.

Proof: Given that L is a lattice and M is a finite chain.

$\therefore M$ is a lattice.

$\psi : L \to M$ is a onto homomorphism.

Let $0'$ be the least element of M .

To show $\ker\psi$ is a prime ideal. First we show that $\ker\psi$ is an ideal.

Let $0'$ be the least element of M.

Since ψ is onto, so

$\exists\ x \in L$ such that $\psi(x) = 0'$

$\Rightarrow x \in \ker\psi$

Therefore $\ker\psi \neq \varphi$

Let $\alpha, \beta \in \ker\psi \Rightarrow \psi(\alpha) = 0', \psi(\beta) = 0'$

Now, $\psi(\alpha \vee \beta) = \psi(\alpha) \vee \psi(\beta)$ [$\because$ ψ is homomorphism]

$= 0' \vee 0'$

$= 0'$

$\Rightarrow \alpha \vee \beta = \ker\psi$

Next $\forall \alpha \in \ker\psi$ and $l \in L$

$\psi(\alpha \wedge l) = \psi(\alpha) \wedge \psi(l)$ [$\because \psi$ is homomorphism]

$= 0' \wedge \psi(l)$

$= 0'$

$\therefore a \wedge l \in \ker\psi$

Hence $\ker\psi$ is an ideal of L.

Let $\alpha \wedge \beta \in \ker\psi$. We have to show that $\alpha \in \ker\psi$ or $\beta \in \ker\psi$.

Now, $\alpha \wedge \beta \in \ker\psi \Rightarrow \psi(\alpha \wedge \beta) = 0'$

$\Rightarrow \psi(\alpha) \wedge \psi(\beta) = 0'$ …..(1)

Since M is a chain so either $\psi(\alpha) \leq \psi(\beta)$ or $\psi(\beta) \leq \psi(\alpha)$

If $\psi(\alpha) \leq \psi(\beta)$ then form (1)

$\psi(\alpha) = 0'$

$\Rightarrow \alpha \in \ker\psi$

Again, if $\psi(\beta) \leq \psi(\alpha)$ then form (1).

$\psi(\beta) = 0'$

$\Rightarrow \beta \in \ker\psi$

So, $\alpha \wedge \beta \in \ker\psi \Rightarrow$ either $\alpha \in \ker\psi$ or $\beta \in \ker\psi$.

Hence $\ker\psi$ is a prime ideal of L. ■

CHAPTER 3

Distributive Lattices

3.1 Introduction:

Distributive lattices have provided the motivation for many results in general lattice theory. In many applications the condition of distributivity is imposed on lattices arising in various areas of mathematics especially algebra. Therefore a thorough knowledge of distributive lattices is indispensable for work in lattice theory. In this chapter we discuss modular and distributive lattices. We also proved the ideal lattice $I(L)$ of a distributive lattice L is distributive iff L is distributive.

3.2 Modularity:

Definition (Modular lattice): A lattice $(L,\leq)$ is called a modular lattice if $\forall x, y, z \in L,$ with $x \geq y$;

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = [y \vee (x \wedge z)]$$

Remark: i) If in the above definition $x = y$, we find

$$x \wedge (y \vee z) = x \wedge (x \vee z) = x$$

$$y \vee (x \wedge z) = x \vee (x \wedge z) = x$$

i.e., the postulate is automatically satisfied.

Example 3.2.1: The lattices given by the following diagrams are modular

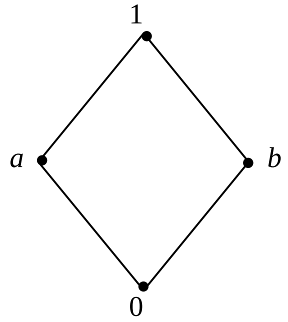


Fig: 3.1

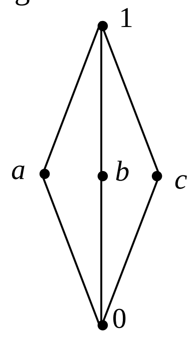


Fig: 3.2

In the first we cannot find any triplet a,b,c s.t., $a>b$ and c is not comparable with a or b. Hence by the remarks above it is modular. By similar argument the second lattice is also seen to be modular.

Example 3.2.2: A chain is a modular lattice, by similar argument.

Definition (Semimodular Lattice): A finite lattice $(L,\leq)$ is called a semimodular lattice if $\forall x, y, \in L,$ the following condition hold:

$$r(x)+r(y)\geq r(x\wedge y)+r(x\vee y)$$

where r is a rank function.

Example 3.2.3: A lattice without finite chain is semimodular lattice.

Theorem 3.2.4: A lattice L is modular of finite length, $l(x)+l(y)=l(x\vee y)+l(x\wedge y)$ where $l(x)$ is the length of the element x.

Proof: Since L is modular lattice so for any

$$x,y\in L[x,x\vee y]\cong[x\wedge y,y]$$

$$\text{Thus, } l([x,x\vee y])=l([x\wedge y,y])$$

$$\Rightarrow l(x\vee y)-l(x)=l(y)-l(x\wedge y)$$

$$\Rightarrow l(x\vee y)+l(x\wedge y)=l(x)+l(y)$$

$$\therefore l(x)+l(y)=l(x\vee y)+l(x\wedge y).\ \blacksquare$$

Theorem 3.2.5: Prove that $N\times N$ is modular, where N is the chain of naturals under usual '$\leq$'.

Proof: Let N be a chain of natural numbers under usual '$\leq$'. Now, we show that N is modular.

Let x,y,z be any three elements of N with $y\leq x$

$\therefore x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$= y \vee (x \wedge z) \qquad [\because y \leq x, x \wedge y = y]$

$= N$ is modular.

Last of all, we have to show that $N \times N$ is modular, let

$(x_{1,} y_1), (x_2, y_2), (x_3, y_3) \in N \times N$ with $(x_2, y_2) \leq (x_1, y_1)$

$$\therefore (x_1, y_1) \wedge [(x_2, y_2) \vee (x_3, y_3)] = (x_1, y_1) \wedge [(x_2 \vee x_3) \vee (y_2 \vee y_3)]$$

$$= (x_1 \wedge (x_2 \vee x_3), y_1 \wedge (y_2 \vee y_3))$$

$$= [x_2 \vee (x_1 \wedge x_3), y_2 \vee (y_1 \wedge y_3)]$$

$$= (x_2, y_2) \vee [(x_1 \wedge x_3), (y_1 \wedge y_3)]$$

$$= (x_2, y_2) \vee [(x_1, y_1), (x_3, y_3)]$$

$= N \times N$ is modular. ■

Theorem 3.2.6: Homomorphic image of a modular lattice is modular.

Proof: Let $\psi : L \to M$ be an onto homomorphism and suppose L is modular.

Let $x, y, z \in M$ be three elements with $x > y$.

Since ψ is onto homomorphism, $\exists$ $p, q, r \in L$ s.t., $\psi(p) = x, \psi(q) = y, \psi(r) = z$ where $p > q$.

Now L is modular, $p, q, r \in L, p > q$, thus we get

$$p \wedge (q \vee r) = q \vee (p \wedge r)$$

Now $\quad \mathrm{x} \wedge (\mathrm{y} \vee \mathrm{z}) = \psi(\mathrm{p}) \wedge (\psi(\mathrm{q}) \vee \psi(\mathrm{r}))$

$$= \psi(p) \wedge (\psi(q \vee r)) = \psi(p \wedge (q \vee r))$$

$$= \psi(q \vee (p \wedge r)) = \psi(q) \vee \psi(p \wedge r)$$

$$= \psi(q) \vee [\psi(p) \wedge \psi(r)] = y \vee (x \wedge z)$$

Hence M is modular. ■

Theorem 3.2.7: Two lattice L_1 and L_2 are modular iff $L_1 \times L_2$ is modular.

Proof: Let $(p_1, q_1), (p_2, q_2), (p_3, q_3) \in L_1 \times L_2$ be three elements with $(p_1, q_1) \geq (p_2, q_2)$

Then $\quad p_1, p_2, p_3 \in L_1, \quad p_1 \geq p_2$

$\quad q_1, q_2, q_3 \in L_2, \quad q_1 \geq q_2$

and since L_1 and L_2 are modular, we get

$$p_1 \wedge (p_2 \vee p_3) = p_2 \vee (p_1 \wedge p_3)$$

$$q_1 \wedge (q_2 \vee q_3) = q_2 \vee (q_1 \wedge q_3)$$

Thus

$$\begin{aligned}
(p_1, q_1) \wedge [(p_2, q_2) \vee (p_3, q_3)] &= (p_1, q_1) \wedge (p_2 \vee p_3, q_2 \vee q_3) \\
&= (p_1 \wedge (p_2 \vee p_3), q_1 \wedge (q_2 \vee q_3)) \\
&= (p_2 \vee (p_1 \wedge p_3), q_2 \vee (q_1 \wedge q_3)) \\
&= (p_2, q_2) \vee (p_1 \wedge p_3, q_1 \wedge q_3) \\
&= (p_2, q_2) \vee [(p_1, q_1) \wedge (p_3, q_3)]
\end{aligned}$$

Hence $L_1 \times L_2$ is modular.

Conversely, let $L_1 \times L_2$ be modular.

Let $\quad p_1, p_2, p_3 \in L_1, \; p_1 \geq p_2$

$\quad q_1, q_2, q_3 \in L_2, \; q_1 \geq q_2$

then $(p_1, q_1), (p_2, q_2), (p_3, q_3) \in L_1 \times L_2$ and $(p_1, q_1) \geq (p_2, q_2)$.

Since $L_1 \times L_2$ is modular, we find

$$(p_1, q_1) \wedge [(p_2, q_2) \vee (p_3, q_3)] = (p_2, q_2) \vee [(p_1, q_1) \wedge (p_3, q_3)]$$

$$\Rightarrow \quad (p_1, q_1) \wedge (p_2 \vee p_3, q_2 \vee q_3) = (p_2, q_2) \vee (p_1 \wedge p_3, q_1 \wedge q_3)$$

$$\Rightarrow \quad (p_1 \wedge (p_2 \vee p_3), q_1 \wedge (q_2 \vee q_3)) = (p_2 \vee (p_1 \wedge p_3), q_2 \vee (q_1 \wedge q_3))$$

$$\Rightarrow \quad p_1 \wedge (p_2 \vee p_3) = p_2 \vee (p_1 \wedge p_3)$$

$$q_1 \wedge (q_2 \vee q_3) = q_2 \vee (q_1 \wedge q_3)$$

$\Rightarrow L_1$ and L_2 are modular. ■

Theorem 3.2.8: A lattice L is modular iff $I(L)$, the ideal lattice of L is modular.

Proof : Let L be modular.

Let $P, Q, R \in I(L)$ be three members s.t., $Q \subseteq P$.

We show $P \cap (Q \vee R) = Q \vee (P \cap R)$

Let $x \in P \cap (Q \vee R)$ be any element.

Then $x \in P$ and $x \in Q \vee R$.

$\Rightarrow$ $x \in P$ and $x \le q \vee r$ for some $q \in Q, r \in R$

Since $q \in Q \subseteq P$, $x \vee q \in P$. Let $x \vee q = p$

Now $x \le q \vee r$, $x \le p \Rightarrow x \le p \wedge (q \vee r)$

$\Rightarrow x \le q \vee (p \wedge r)$ as $p \ge q$ and L is modular.

Again, $p \wedge r \le p$, $p \in P \Rightarrow p \wedge r \in P$

$p \wedge r \le r$, $r \in R \Rightarrow p \wedge r \in R$

Thus $p \wedge r \in P \cap R$ and as $q \in Q$ we find $x \in Q \vee (P \cap R)$

i.e. $P \cap (Q \vee R) \subseteq Q \vee (P \cap R)$

$Q \vee (P \cap R) \subseteq P \cap (Q \vee R)$ follows by modular inequality, or to prove it independently, let $y \in Q \vee (P \cap R)$.

Then $y \le q \vee k$ where $q \in Q$, $k \in P \cap R$

Thus $y \le q \vee k$, $(q \in Q \subseteq P, k \in P \Rightarrow q \vee k \in P)$

$\Rightarrow$ $y \in P$

Also $y \le q \vee k$, $q \in Q$, $k \in R \Rightarrow y \in Q \vee R$

i.e., $y \in P \cap (Q \vee R)$

Showing that $Q \vee (P \cap R) \subseteq P \cap (Q \vee R)$

Hence $P \wedge (Q \cup R) = Q \vee (P \cap R)$ or that $I(L)$ is modular.

Conversely, let $I(L)$ be modular, Since L can be imbedded into $I(L)$, it is isomorphic to a sublattice of $I(L)$. This sublattice must be modular as $I(L)$ is modular. Hence L is modular. ■

Theorem 3.2.9: Any non modular lattice L contains a sublattice isomorphic with the pentagonal lattice.

Proof: Since L is non modular $\exists$ at least three elements a,b,c $a \geq b$ s.t., $a \wedge (b \vee c) \neq b \vee (a \wedge c)$.

In view of the remarks of definition, we must have $a > b,$ and as in any lattice the modular lattice inequality $(a \geq b, a \wedge (b \vee c) \geq b \vee (a \wedge c))$ holds.

we get $\quad a \wedge (b \vee c) > b \vee (a \wedge c).$

Consider the chain

$$a \wedge c \leq b \vee (a \wedge c) < a \wedge (b \vee c) \leq b \vee c \text{ ....(1)}$$

We show at all place, strict inequality holds.

Suppose $\quad a \wedge c = b \vee (a \wedge c)$

Then $b \leq a \wedge c \qquad (x = y \vee x \Rightarrow y \leq x)$

$\Rightarrow \qquad b \vee c \leq (a \wedge c) \vee c$

$\Rightarrow \qquad b \vee c \leq c \leq b \vee c$

$\Rightarrow \qquad b \vee c = c$

$\Rightarrow \qquad a \wedge (b \vee c) = a \wedge c,$ $\qquad$ a contradiction to (1)

Thus $a \wedge c < b \vee (a \wedge c).$ Similarly $a \wedge (b \vee c) < b \vee c$.

Hence chain (1) becomes

$$a \wedge c < b \vee (a \vee c) < a \wedge (b \vee c) < b \vee c \text{ ....(2)}$$

Consider now the chain

$$a \wedge c \leq c \leq b \vee c$$

As seen above $b \vee c = c$ leads to a contradiction and similarly $a \wedge c = c$ would give a contradiction.

Hence $\quad a \wedge c < c < b \vee c$ ....(3)

We thus have two chains (2) and (3) with same end points.

We show c does not lie in chain (2). For this it is sufficient to prove that c is not comparable with $a \wedge (b \vee c)$.

Suppose $\quad a \wedge (b \vee c) \leq c$

Then $a \wedge (a \wedge (b \vee c)) \leq a \wedge c$

$\Rightarrow \qquad a \wedge (b \vee c) \leq a \wedge c$ $\qquad$ a contradiction to (2)

Again, if $\quad a \wedge (b \vee c) > c$

then as $\quad a \geq a \wedge (b \vee c)$

We find $\quad a > c$ which gives $a \wedge c = c,$ a contradiction to (3)

Hence the chain (2) and (3) form a pentagonal subset

$S = \{a \wedge c, b \vee (a \wedge c), a \wedge (b \vee c), b \vee c, c\}$ of L.

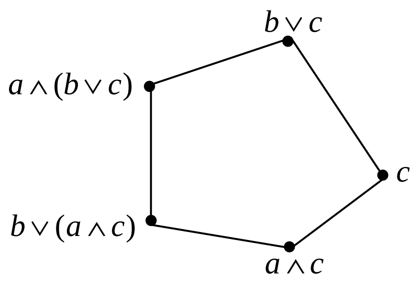


Fig. 3.3

We show now this pentagonal subset is a sublatice. For that meet and join of any two elements of S should lie inside S. Meet and join of any two comparable elements being one of them is clearly in S.

Now $\left[a \wedge (b \vee c)\right] \wedge c = a \wedge \left[(b \vee c) \wedge c\right] = a \wedge c \in S$

Also $\left[a \wedge (b \vee c)\right] \vee c \geq \left[b \vee (a \wedge c)\right] \vee c$ by (2)

$$= b \vee [(a \wedge c) \vee c] = b \vee c$$

and $a \wedge (b \vee c) \leq b \vee c$ gives

$$\left(a \wedge (b \vee c)\right) \vee c \leq (b \vee c) \vee c = b \vee c$$

Thus $\left[a \wedge (b \vee c)\right] \vee c = b \vee c \in S.$

Similarly, we can show $\left[b \vee (a \wedge c)\right] \vee c = b \vee c \in S$

$$\left[b \vee (a \wedge c)\right] \wedge c = a \wedge c \in S$$

Hence S forms a sublattice of L. ■

3.3 Distributive Lattice and its related theorems:

Definition (Distributive lattice): A lattice $(L, \leq)$ is called a distributive lattice if and only if the distributive laws hold; that is, for all $x, y, z \in L$, we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Example 3.3.1: If X is any set, then the lattice $(p(X), \leq)$ is a distributive lattice.

Remarks: For a distributive Lattice L, $J(L)$ denotes the set of all nonzero join irreducible elements, regarded as a poset under the partial ordering of L. $H(J(L))$ denotes the set of all hereditary subsets partially ordered by set inclusion. $H(J(L))$ is a Lattice in which meet & join are intersection & union respectively. Hence $H(J(L))$ is a distributive Lattice for $a \in L$, set $r(a) = \{x \in J(L) | x \le a\}$.

Example 3.3.2: We now give an example of a lattice where the distributive laws do not hold. Let $L = \{1, 2, 3, 5, 30\}$. Then L is a poset under the relation divides. The operation table for $\wedge$ and $\vee$ on L are:

$\vee$	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

$\wedge$	1	2	3	5	30
1	1	1	1	1	1
2	2	1	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

Since every pair of elements in L has both a join and a meet, so L is a lattice (under divides). But

$$2 \vee (5 \wedge 3) = 2 \vee 1 = 2 \text{ and}$$

$$(2 \vee 5) \wedge (2 \vee 3) = 30 \wedge 30 = 30$$

so that $x \vee (y \wedge z) \ne (x \vee y) \wedge (x \vee z)$ for some values of $x, y, z \in L$. Hence L is not distributive lattice.

Theorem 3.3.3: Let L be a finite distributive lattice. Then the map $\psi : a \to r(a)$ is a isomorphism between L and $H(J(L))$.

Proof: Define $\psi : L \to H(J(L))$ by $\psi(a) = r(a), a \in L$.

Since L is finite, so every element is the join of join irreducible elements. Thus $a \in L \Rightarrow a = \vee r(a)$.

Obviously $\psi(a \wedge b) = \psi(a) \cap \psi(b)$. So ψ is a meet homomorphism. To show that ψ is a join homomorphism. We are to show that $r(a \vee b) = r(a) \cup r(b)$.

Now $r(a) \cup r(b) \subseteq r(a \vee b)$ is obvious.

Let $x \in r(a \vee b)$

$\Rightarrow x \le a \vee b$

$\Rightarrow x = x \wedge (a \vee b)$

$$=(x\wedge a)\vee(x\wedge b)$$

Since $x\in J(L)$, so we have either $x=x\wedge a$ or $x=x\wedge b$

$\Rightarrow$ either $x\le a$ or $x\le b$

$\Rightarrow$ either $x\in r(a)$ or $x\in r(b)$

$\Rightarrow x\in r(a)\cup r(b)$

Hence, $r(a\vee b)\subseteq r(a)\cup r(b)$.

Therefore, $r(a\vee b)=r(a)\cup r(b)$. So ψ is a join homomorphism.

Therefore, ψ is a homomorphism.

Suppose $\psi(a)=\psi(b),\ \ a,b\in L$

$$\Rightarrow r(a)=r(b)$$
$$\Rightarrow \vee r(a)=\vee r(b)$$
$$\Rightarrow a=b$$

Hence ψ is one-one.

To show ψ is onto. Let $A\in H(J(L))$ and $a\in L$. Set $a=\vee A$. We are to show that $r(a)=A$.

Clearly, $A\subseteq r(a)$.

Let $x\in r(a)\Rightarrow x\le a$

$$\Rightarrow x=x\wedge a$$
$$=x\wedge(\vee A)$$
$$=\vee(x\wedge t\,|\,t\in A)\text{(since } L \text{ is distributive)}$$

Since $x\in J(L)$ so $x=x\wedge t$ for some $t\in A.$

$$\Rightarrow x\le t$$
$$\Rightarrow x\in A \text{ as } A\in H(J(L))$$
$$\Rightarrow r(a)\subseteq A$$
$$\therefore r(a)=A$$
$$\Rightarrow \psi(a)=A$$

Hence ψ is onto.

Therefore, $L\cong H(J(L))$. ■

Theorem 3.3.4: Prove that a distributive lattice is always modular but converse is not true.

Proof: Let, L is a distributive lattice, $a,b,c\in L$ with $c\le a$

Then, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$

$$=(a\wedge b)\vee c$$

Thus L is modular.

For the converse, consider the lattice

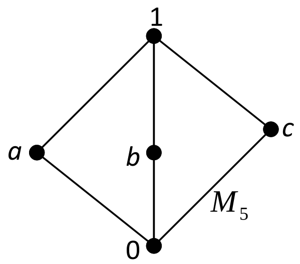


Fig-3.4

It is easy to check that M_5 is modular.

But in M_5, $a \wedge (b \vee c) = a \wedge 1 = a$

$$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$$

i.e., $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Therefore, L is not distributive lattice. ■

Theorem 3.3.5: A lattice L is distributive if and only if for any two ideal I and J of L,

$$I \vee J = \{i \vee j : i \in I, j \in J\}$$

Proof: First suppose a modular lattice L is distributive. Let $I, J \in I(L)$. Then for $x \in I \vee J$ implies that $x \leq i \vee j$ for some $i \in I, j \in J$.

Then $x = x \wedge (i \vee j)$

$= (x \wedge i) \vee (x \wedge j)$

Since L is distributive, when $x \wedge i \in I$ and $x \wedge j \in J$

Therefore, $I \vee J = \{i \vee j : i \in I, j \in J\}$

Conversely, suppose that $I \vee J = \{i \vee j : i \in I, j \in J\}$ for any two ideals I and J of L. We are to show that L is distributive.

Suppose L is not distributive, then it has a sublattice isomorphism to M_5 or N_5.

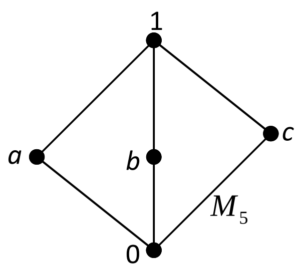


Fig-3.5

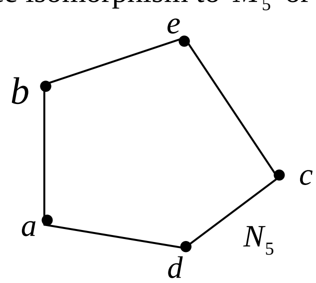


Fig. 3.6

Here observe that in both cases $b \in (a] \vee (c]$, but $b \neq i \vee j$ for any $i \in (a]$ and $j \in (c]$.

Hence L is distributive. ■

Theorem 3.3.6: A lattice L is distributive iff

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad \forall x, y, z \in L .$$

Proof : Let L be distributive.

Now $(x \vee y) \wedge (x \vee z) = [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z]$

$= x \vee [(x \vee y) \wedge z]$

$= x \vee [(x \wedge z) \vee (y \wedge z)]$

$= [x \vee (x \wedge z)] \vee (y \wedge z)$

$= x \vee (y \wedge z)$

Conversely, let $x, y, z \in L$ be any three elements, then

$(x \wedge y) \vee (x \wedge z) = [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z]$

$= x \wedge [(x \wedge y) \vee z]$

$= x \wedge [(z \vee x)] \wedge (z \vee y)]$

$= [x \wedge (z \vee x)] \wedge (z \vee x)$

$= x \wedge (y \vee z) = x \wedge (y \vee z)$

i.e., L is distributive. ■

Note: Dual of a distributive lattice is distributive.

Theorem 3.3.7: A lattice L is distributive if and only if $I(L)$ is distributive; $I(L)$ is the set of all ideals.

Proof: Suppose L is distributive. Let $P, Q, R \in I(L)$. We need to show that

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) .$$

The relation $(P \wedge Q) \vee (P \wedge R) \subseteq P \wedge (Q \vee R)$ is obviously true. Let $x \in P \wedge (Q \vee R),$ then $x \in P$ and $x \in Q \vee R$. Since L is distributive.

So $x = x \wedge (q \vee r) = (x \wedge q) \vee (x \wedge r) \in (P \wedge Q) \vee (P \wedge R)$ for some $q \in Q, r \in R.$

Then, $P \wedge (Q \vee R) \subseteq (P \wedge Q) \vee (P \wedge R)$

$\therefore P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

$\therefore I(L)$ is distributive.

Conversely, suppose, $I(L)$ is distributive. Let $x, y, z \in l$, then

$(x \wedge (y \vee z)] = (x] \wedge (y \vee z]$

$= (x] \wedge [(y] \vee (z])$ as $I(L)$ is distributive.

$= (x \wedge y] \vee (x \wedge z]$

$$= ((x \wedge y) \vee (x \wedge z)]$$

$$\Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

so L is distributive. ■

Theorem 3.3.8: A lattice L is distributive iff

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \ \forall a, b, c \in L .$$

Proof: Let L be a distributive lattice.

$$\begin{aligned}(a \vee b) \wedge (b \vee c) \wedge (c \vee a) &= \{a \wedge [(b \vee c) \wedge (c \vee a)]\} \vee \{b \wedge [(b \vee c) \wedge (c \vee a)]\} \\ &= [\{a \wedge (c \vee a)\} \wedge (b \vee c)] \vee [\{b \wedge (b \vee c)\} \wedge (c \vee a)] \\ &= [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)] \\ &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \vee (b \wedge a) \\ &= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)\end{aligned}$$

Conversely, we first show that L is modular.

Let x, y, z be any three elements of L, with $x \geq y$

Then

$$\begin{aligned}x \wedge (y \vee z) &= [x \wedge (x \vee z)] \wedge (y \vee z) && \text{(absorption)} \\ &= (x \vee y) \wedge (x \vee z) \wedge (y \vee z) && (x \geq y) \\ &= (x \vee y) \wedge (y \vee z) \wedge (z \vee x) \\ &= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \\ &= (y \vee (y \wedge z)) \vee (z \wedge x) && (x \geq y) \\ &= y \vee (x \wedge z)\end{aligned}$$

i.e., L is modular.

Now for any $a, b, c \in L$

$$\begin{aligned}a \wedge (b \vee c) &= [a \wedge (a \vee c)] \wedge (b \vee c) \\ &= [a \wedge (a \vee b) \wedge (a \vee c) \wedge (b \vee c)] \\ &= a \wedge [(a \vee b) \wedge (b \vee c) \wedge (c \vee a)] \\ &= a \wedge [(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)] \\ &= a \wedge [(b \wedge c) \vee ((a \wedge b) \vee (c \wedge a))]\end{aligned}$$

Now using modularity, as $a \geq a \wedge b,\ a \geq c \wedge a$ gives $a \geq (a \wedge b) \vee (c \wedge a)$ we get

$$a \wedge (b \vee c) = [(a \wedge b) \vee (c \wedge a)] \vee [(b \wedge c) \wedge a]$$

$$= (a \wedge b) \vee [(c \wedge a) \vee [(c \wedge a) \wedge b)]$$

$$= (a \wedge b) \vee (c \wedge a)$$

Hence L is distributive. ■

Theorem 3.3.9: Let L be a distributive and $a \in L$, the map $\psi : x \rightarrow \langle x \wedge m, x \vee m \rangle$ is an embedding of L into $(m] \times [m)$. It is an isomorphism if m has a complement.

Proof: For $x, y \in L$

we have, $\psi(x) = \langle x \wedge m, x \vee m \rangle$ and $\psi(y) = \langle y \wedge m, y \vee m \rangle$

Then $\psi(x \wedge y) = \langle (x \wedge y) \wedge m, (x \wedge y) \vee m \rangle$

$$= \langle x \wedge y \wedge m, (x \vee m) \wedge (y \vee m) \rangle$$

$$= \langle (x \wedge m) \wedge (y \wedge m), (x \vee m) \wedge (y \vee m) \rangle$$

$$= \langle x \wedge m, x \vee m \rangle \wedge \langle y \wedge m, y \vee m \rangle$$

$$= \psi(x) \wedge \psi(y)$$

and $\psi(x \vee y) = \langle (x \vee y) \wedge m, (x \vee y) \vee m \rangle$

$$= \langle (x \wedge m) \vee (y \wedge m), (x \vee m) \vee (y \vee m) \rangle$$

$$= \langle x \wedge m, x \vee m \rangle \vee \langle y \wedge m, y \vee m \rangle$$

$$= \psi(x) \vee \psi(y)$$

Hence ψ is a homomorphism.

Now let $\psi(x) = \psi(y), x, y \in L.$

Then $\langle x \wedge m, x \vee m \rangle = \langle y \wedge m, y \vee m \rangle$ and

So, $x \wedge m = y \wedge m$ and $x \vee m = y \vee m$

Now, $x = x \wedge (x \vee m)$

$$= x \wedge (y \vee m)$$

$$= (x \wedge y) \vee (x \wedge m)$$

$$= y \wedge (x \vee m)$$

$$= y \wedge (y \vee m)$$

$$= y$$

$\Rightarrow x = y$ and so ψ is one-one.

Hence ψ is an embedding.

2nd part: Let $m \in L$ has a complement. Choose an element $\langle x, y \rangle \in (m] \times [m)$, then $x \leq m \leq y$. Since m has a complement in L so it has a relative complement n in the interval $[x, y]$

Then we have, $m \wedge n = x$ and $m \vee n = y$

$$\therefore \langle x, y \rangle = \langle m \wedge n, m \vee n \rangle$$

$= \psi(n)$

Hence ψ is onto. Therefore ψ is an isomorphism. ■

Theorem 3.3.10: Homomorphic image of a distributive lattice is distributive.

Proof : Let $\psi : L \to M$ be an onto homomorphism where L is distributive lattice.

Let $x, y, z \in M$ be any elements. Since ψ is onto,

$\exists\ a, b, c \in L$ s.t., $\psi(a) = x,\ \psi(b) = y,\ \psi(c) = z$

Now $x \wedge (y \vee z) = \psi(a) \wedge [\psi(b) \vee \psi(c)]$

$= \psi(a) \wedge (\psi(b \vee c))$

$= \psi(a \wedge (b \vee c))$

$= \psi((a \wedge b) \vee (a \wedge c))$

$= \psi(a \wedge b) \vee \psi(a \wedge c)$

$= (\psi(a) \wedge \psi(b)) \vee (\psi(a) \wedge \psi(c))$

$= (x \wedge y) \vee (x \wedge z)$

Therefore M is distributive. ■

Theorem 3.3.11: For any two ideals *I* and *J* of a distributive lattice *L* if $I \wedge J$ and $I \vee J$ are principal then both *I* and *J* are principal.

Proof: Let $I \wedge J = (x]$ and $I \vee J = (y]$

Then $y = i \vee j$ for some $i \in I$ and $j \in J$. Set $c = x \vee i$ and $b = x \vee j$

Then clearly $c \in I$ and $b \in J$

We have to show that $I = (c]$ and $J = (b]$

If $I \neq (c]$, then there exists and element $a > c$ such that $a \in I$

Moreover, the set $\{x, a, b, c, y\}$ form a lattice isomorphic to N_5

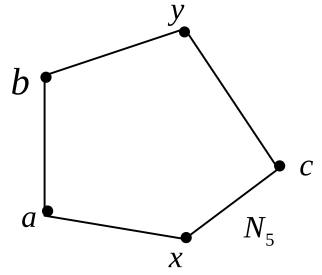


Fig. 3.7

i.e., L is not distributive. This is a contradiction
Hence $I=(c]$. Therefore I is a principal ideal.
Similarly, we can show that, $J=(b]$, i.e., J is also a principal ideal.Hence proved. ■

Theorem 3.3.12: A modular lattice is disributive if and only if it has no sublattice isomorphic M_5.

Proof: First suppose a modular lattice L is distributive. Then its every sublattice is also distributive.

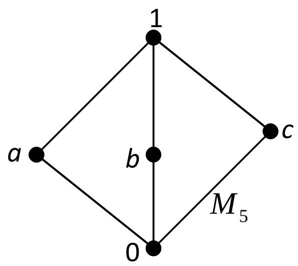


Fig-3.8

Since M_5 is not distributive (For $a\wedge(b\vee c)=a\wedge 1=a$ but $(a\wedge b)\vee(a\wedge c)=0\vee 0=0$) So, L can not contain any sublattice isomorphic to M_5.
Conversely, suppose that L is not distributive. Then there exist elements $x,y,z\in L$ such that $x\wedge(y\vee z)\neq(x\wedge y)\vee(x\wedge z)$ but

$$(x\wedge y)\vee(x\wedge z)\leq x\wedge(y\vee z)$$
$$\Rightarrow (x\wedge y)\vee(x\wedge z)< x\wedge(y\vee z)$$

Thus every modular lattice which is not distributive contains a sublattice isomorphic to M_5.
Hence L is a distributive. ■

Theorem 3.3.13: The ideal lattice $I(L)$ of a distributive lattice L is distributive iff L is distributive.

Proof: Let L be distributive.

Let $P,Q,R\in I(L)$ be any three members, then P,Q,R are ideals of L, We show

$$P\wedge(Q\vee R)=(P\wedge Q)\vee(P\wedge R)$$

i.e., $$P\cap(Q\vee R)=(P\cap Q)\vee(P\cap R)$$

Let $x\in P\cap(Q\vee R)$ be any element.

Then $x\in P$ and $x\in Q\vee R$

$\Rightarrow\ \exists\, q\in Q, r\in R$ s.t., $x\le q\vee r$

$\Rightarrow\ x\wedge(q\vee r)=x$

$\Rightarrow\ (x\wedge q)\vee(x\wedge r)=x$ (as L is distributive)

Now $x\in P, q\in Q\subseteq L\Rightarrow x\wedge q\in Q$

Again, $x\in P\subseteq L$ and $q\in Q\Rightarrow x\wedge q\in Q$

$\Rightarrow\qquad (x\wedge q)\in P\cap Q$.

Similarly, $(x\wedge r)\in P\cap R$

Since $x=(x\wedge q)\vee(x\wedge r)$, by definition of $\vee$ in $I(L)$

We find $x\in(P\cap Q)\vee(P\cap R)$

i.e., $P\cap(Q\vee R)\subseteq(P\cap Q)\vee(P\cap R)$

Again, let $x\in(P\cap Q)\vee(P\cap R)$ be any element.

Then $x\le k_1\vee k_2$ for some $k_1\in P\cap Q$, $k_2\in P\cap R$.

Now $k_1\vee k_2\in P$, $x\le k_1\vee k_2$ thus $x\in P$

Also $k_1\in Q$, $k_2\in R$ and $x\le k_1\vee k_2\Rightarrow x\in Q\vee R$.

Thus $x\in P\cap(Q\vee R)$

Or that $\qquad (P\cap Q)\vee(P\cap R)\subseteq P\cap(Q\vee R)$

i.e., $\qquad P\cap(Q\vee R)=(P\cap Q)\vee(P\cap R)$

and hence $I(L)$ is distributive.

Conversely, since $\exists$ 1-1 homomorphism from $L\to I(L)$, L will be isomorphic to a sublattice of $I(L)$. If $I(L)$ is distributive, this sublattice and hence L will be distributive. Thus converse also holds. ■

3.4 Atomic Lattice:

Definition (Atom): An element in a poset which covers 0 is called an atom.

Definition (Dual Atom): An element in a poset which is covered by 1 the greatest element of the poset is called an dual-atom.

Definition (Atomic Lattice): An atomic lattice is one in which each element other than 0 includes at least one atom.

Example 3.4.1: Any power set $P(A)$ of the set A is an atomic lattice since the one element subsets of A are clearly atoms of $P(A)$ and every subset of A excepting the void subset, includes at least one atom.

Theorem 3.4.2: In an atomic lattice in which each element has a unique complement two elements are equal if and only if they contain the same atoms.

Proof: Suppose that x and z contains exactly the same atoms. That is to say

$$P \leq x \Leftrightarrow P \leq z$$

Now if $P \leq x$ then $P \leq z \Rightarrow P \leq x \wedge z$

Now, $P \leq x$ and $P \leq x \wedge z$ shows that

x and $x \wedge z$ contains exactly the same atoms.

But if $x \neq z$, then either

$x \wedge z < x$ or $x \wedge z < z$

Take, $x \cap z = y$

Now, $x > x \cap z = y \Rightarrow y' \cup x \geq y' \cup y = 1$

Then $y' \cap x \neq 0$, for otherwise if $y' \cap x = 0$ then

$y' \cup x = 1$ and $y' \cap x = 0$

Also $y' \cup y = 1$ and $y' \cap y = 0$, gives

$x = y$

So there exists and atom P such that

$$P \le y' \cap x$$

$\Rightarrow P \le x$ and $P \le y'$

$\Rightarrow P \le x$ and $P \cap y \le y' \cap y = 0$

and $P \nleq y = x \cap z$

If $x \cap z < x$ then x contains an atom P not contained in $x \cap z$. The contradication shows that the supposition $x \ne z$ is false. Since the argument in the other case is similar. We see that if x and z contain the same atoms then $x = z$

Conversely, let x and z be two equal elements. We have to show that they contain same atoms.

If possible let p and q are two distinct atoms of x and y respectively. Then p is atom of $x \Rightarrow p \le x = z$

$\Rightarrow p \le z$

Again, q is atom of $z \Rightarrow q \le z$

$$\Rightarrow p \le q \le z = x$$

$$\Rightarrow p \le q \le x$$

which shows that p is not atom of x.

a contradiction, thus $p = q$. ■

CHAPTER 4

Boolean function and its different forms

4.1 Introduction:

A complemented distributive lattice is called a Boolean Lattice. Let $(B,\wedge,\vee,',0,1)$ be a Boolean. Expressions involving member of B and the operations $\wedge,\vee$ and complementation are called Boolean expression. Any function specifying these Boolean expressions is called a Boolean function. A Boolean function is said to be in Disjunctive normal form (DN form) in n variables $x_1,x_2,x_3,..................,x_n$ if it can be written as join of terms of the type $f_1(x_1)\wedge f_2(x_2)\wedge f_3(x_3)\wedge..........\wedge f_n(x_n)$, where $f_i(x_i)=x_i$ for all $i=1,2,3,............,n$ and no two terms are same. A Boolean function f is said to be in Conjunctive Normal Form (CN from) in n variables $x_1,x_2,x_3...........,x_n$ if f *is* meet of terns of the type $f_1(x_1)\vee f_2(x_2)\vee....................\vee f_n(x_n)$ where $f_i(x_i)=x_i$ or x_i' for all $i=1,2,3,..........,n$ and no two terms are same.

4.2 Boolean function:

Definition (Boolean lattice and Boolean Algebra) : A complemented distributive lattice is called a Boolean Lattice. Since complements are unique in a Boolean Lattice we can regard a Boolean Lattice as an algebra with two binary operations $\wedge$ and $\vee$ and one unary operation $'$. Boolean Lattice so considered is called Boolean Algebra. In other words, by a Boolean Algebra, we mean a system consisting of a non empty set L together with two binary operations $\wedge$ and $\vee$ and unary operation $'$, 0 and 1 satisfying $(\forall,a,b,c\in L)$.

$(i)\, a\wedge a=a, a\vee a=a$

$(ii)\, a\wedge b=b\wedge a, a\vee b=b\vee a$

$(iii)\, a\wedge(b\wedge c)=(a\wedge b)\wedge c, a\vee(b\vee c)=(a\vee b)\vee c$

$(iv)\, a\wedge(a\vee b)=a, a\vee(a\wedge b)=a$

$(v) a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$(vi) \forall a \in L, \exists\ a' \in L,$ s.t., $a \wedge a' = 0\ \ a \vee a' = 1$

where 0,1 are elements of L satisfying $0 \le x \le 1\ \ \forall x \in L.$

Example 4.2.1: Let $B = \{0, x, y, 1\}$. If we define $\wedge, \vee$ and complementation ´ by

$\wedge$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

$\vee$	0	x	y	1
0	0	x	y	1
x	x	x	1	1
y	y	1	y	1
1	1	1	1	1

´	
0	1
x	y
y	x
1	0

Then B forms a Boolean algebra under these operations. Since a Boolean Algebra is distributive (and thus, modular) and complemented, all properties of modular, distributive and complemented lattices hold in a Boolean algebra.

Definition (Boolean expression): Let $(B, \wedge, \vee, , 0, 1)$ be a Boolean algebra. Then any expression involving members of B and the operation $\wedge, \vee$ and complementation is called a Boolean expression or Boolean Polynomial.

Example 4.2.2: If $x, y \in B$ then $x \vee y,\ x \vee y',\ x \wedge y,\ x' \wedge y'$ etc are Boolean expressions.

Remark: If e_1 and e_2 are Bollean expressions, then e_1', $e_1 \vee e_2$, $e_1 \wedge e_2$ are Bollean expressions.

A Boolean expression that contains n distinct variables is usually referred to as a Bollean expression of n variables.

Definition (Boolean function): Let $(B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. A function $f : B^n \to B$ is called a Boolean function if it can be specified by a Bollean expression of n variables.

Example 4.2.3 : $f(x, y) = x \wedge y$ is a Boolean function.

Remark: $f(x,y)=x\wedge y$ then f is a Boolean function and $x\wedge y$ is the Boolean expression (or value of the function). In fact, the Boolean expressions are the Boolean functions.

Examples 4.2.4: A function $f:\{0,1\}^2\to\{0,1\}$ is defined by

$f(0,0)=0,\ f(0,1)=1,\ f(1,0)=0,\ f(1,1)=0$

Is a Boolean function as the Boolean expression $(x'\wedge y)\vee(x\wedge y')$ over the Boolean algebra $(\{0,1\},\wedge,\vee,')$ defines the function f .

Example 4.2.5 : Every function $f:B^n\to B$ can not be specified by a Boolean expression over $(B,\vee,\wedge,')$. For example, in the following function

$f:\{0,1,2,3\}^2\to\{0,1,2,3\}$ is defined by

$f(0,0)=1$	$f(1,0)=1$	$f(2,0)=2$	$f(3,0)=3$
$f(0,1)=0$	$f(1,1)=1$	$f(2,1)=0$	$f(3,1)=0$
$f(0,2)=0$	$f(1,2)=0$	$f(2,2)=1$	$f(3,2)=0$
$f(0,3)=3$	$f(1,3)=3$	$f(2,3)=1$	$f(3,3)=2$

Fig. 4.1

There is no Boolean expression over the Boolean algebra $(\{0,1,2,3,\},\wedge,\vee,')$ that defines the function in Fig. 4.1 .

Hence $f:\{0,1,2,3\}^2\to\{0,1,2,3\}$ is not a Boolean function.

Theorem (Birkhaf stone theorem) 4.2.6: Let I be an ideal and D be a dual ideal of a distributive lattice L with $I\cap D=\varphi$. Then there exist a prime ideal $P\supseteq I$ such that $P\cap D=\varphi$.

Theorem 4.2.7: Let L be a distributive lattice with 0 and 1. Then L is a Boolean lattice if and only if $P(L)$, the set of all prime ideals of L is unordered.

Proof: First suppose L is a Boolean lattice.

Suppose $P(L)$ is not unordered. Then there exist $P,Q \in P(L)$. Then there exists an $a \in Q-P$. Now $a \wedge a' = 0 \in P$. Since P is prime and $a \notin P$ implies $a' \in P \subset Q$.

$\Rightarrow a' \in Q$.

Thus $a \vee a' = 1 \in Q$. Which is a contradiction as Q is prime.

Hence $P(L)$ is unordered.

Conversely, suppose that $P(L)$ is unordered. We have to show that L is a Boolean lattice.

If L is not Boolean, then there exist an element $a \in L$ which has no complement.

Set $D = \{x \mid a \vee x = 1\}$. Then D is a dual ideal.Consider $D_1 = D \vee [a) = \{x \mid x \geq d \wedge a\}$ for some $d \in D$.$[D = \{x \mid a \vee x = 1\}, [a) = \{x \mid a \leq x\}, D \vee a = \{x \mid x \geq a \geq a \wedge d\}$ for some $d \in D]$

Now we have to show that D_1 does not contain 0.If D_1 contain 0, then $0 = d \wedge a$ for some $d \in D$. Then we have $d \vee a = 1$. Which gives a contradiction as L is not Boolean.Hence $0 \notin D_1$. Then there exists a prime P such that $P \cap D_1 = \varphi$.

Now $1 \notin [a) \vee P$ for otherwise $1 = a \vee p$ for some $p \in P$. Then by stone representration theorem we have there exist a prime ideal Q containing $P \vee (a]$. Thus $P \subset Q$ which is impossible as the set of prime ideals are unordered.

Hence L must be Boolean. ■

4.3 Disjunctive normal form or DN form:

Definition (DN form) : A Boolean function (Expression) is said to be in disjunctive normal form (DN form) in n variables $x_1, x_2, \ldots\ldots, x_n$ if it can be written as join of terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge \ldots\ldots \wedge f_n(x_n)$ Where $f_i(x_i) = x_i$ or x_i', for all $i = 1, 2, \ldots, n$.

Definition (Complete DN form): If number of variables is n, then the total number of minterms will be 2^n. If a disjunctive normal form in n variables contains all the 2^n minterms then it is called the complete disjunctive normal form in n variables.

Definition (Minterm): A Boolean expression of n variables $x_1, x_2, ..., x_n$ is said to be a minterm or minterm polynomial if it is of the form $f_1(x_1) \wedge f_2(x_2) \wedge ...... \wedge f_n(x_n)$, where $f_i(x_i) = x_i$ or x_i', for all $i = 1, 2, ..., n$.

Example 4.3.1 : $(x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z)$

is in disjunctive normal form in there variables x, y and z. Here the terms $(x \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$ and $(x' \wedge y \wedge z)$ are minterms or minterm polynomials.

Remark : i) Thus each minterm is a meet of all the n-variables with or without a prime (complementation operation).

ii) If we have three variables x, y, z then any function in the DN form will be join of some or all the minterms.

$x \wedge y \wedge z, x' \wedge y \wedge z, x \wedge y' \wedge z, x \wedge y \wedge z', x \wedge y' \wedge z', x' \wedge y \wedge z', x' \wedge y' \wedge z, x' \wedge y' \wedge z'$ which will be $2^n (n = 3)$ in number.

Theorem 4.3.2 : Every Boolean function can be put in disjunctive normal form.

Proof: We prove the result by talking the following steps:

i) If primes (complementation operation) occur outside brackets, then open the brackets by using De Morgan's laws,

$(x \wedge y)' = x' \vee y'$; $(x \vee y)' = x' \wedge y'$

ii) Open all brackets by using distributivity and simplify using any of the definition conditions like idempotency, absorption etc.

iii) If any of the terms does not contain a certain variable x_i (or x_i') then take meet of that term with $x_i \vee x_i'$. Do this with each such term (it will not affect the function as $x_i \vee x_i' = 1$ and $1 \wedge x = x$)

Now open brackets and droop all terms of the types $x \wedge x' (= 0)$. Again, if any of the terms occur more than once, these can be omitted because of idempotency. The resulting expression will be in DN form. Hence every function in a Boolean algebra is equal to a function in DN form. ■

Theorem 4.3.3 : Every Boolean function can be expressed in DN form in one and only one way.

Proof: Suppose f is a Boolean function and

Let $f = A_1 \vee A_2 \vee .... \vee A_n$ and $f = B_1 \vee B_2 \vee .... \vee B_m$

(Where A_i and B_i are minterms) be two representations of f in DN form. Then all $A_1, A_2, ..., A_n$ and all $B_1, B_2, ..., B_m$ will be distinct by definition.

In general, if X and Y be two distinct minterms then $X \wedge Y = 0$ as X would always contain at least one x_i such that Y contains x_i'.

Now, $f = A_1 \vee A_2 \vee ...... \vee A_n = B_1 \vee B_2 \vee ...... \vee B_m$

$\Rightarrow A_i \leq B_1 \vee B_2 \vee .... \vee B_m, \forall i = 1, 2, ..., n$.

$\Rightarrow A_i = A_i \wedge (B_1 \vee B_2 \vee ...... \vee B_m)$

$= (A_i \wedge B_1) \vee (A_i \wedge B_2) \vee .... \vee (A_i \wedge B_m), \forall i = 1, 2, ..., n$

Now if A_i does not equal any of $B_1, B_2, ..., B_m$ then the R.H.S. is zero which means $A_i = 0$, But it is not true. Thus A_i equals some B_j (it can not be equal to two or more B_j's as B_j's are all distinct).

Similarly each B_j is equal to same A_i. Hence the two representations of f are same (because of commutativity, the order in which the terms occur is immaterial). We thus conclude, there is once and only one way to write a given Boolean function in the DN form (in a given number of variables). ■

Problem 4.3.4 : Put the function $f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$ in the DN form.

Solution: We have

$$f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$$

$$= [(x' \vee y'') \vee z'] \wedge (x'' \wedge z')$$

$$=[(x' \vee y) \vee z'] \wedge (x \wedge z')$$

$$= (x' \vee y \vee z') \wedge (x \wedge z')$$

$$= (x' \wedge x \wedge z') \vee (y \wedge x \wedge z') \vee (z' \wedge x \wedge z')$$

$$= (0 \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge z')$$

$$= 0 \vee (x \wedge y \wedge z') \vee (x \wedge z')$$

$$= (x \wedge y \wedge z') \vee [(x \wedge z') \wedge (y \vee y')]$$

$$= (x \wedge y \wedge z') \vee [(x \wedge z' \wedge y) \vee (x \wedge z' \wedge y')]$$

$$= (x \wedge y \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z')$$

$= (x \wedge y \wedge z') \vee (x \wedge y' \wedge z')$, Which is the DN form of f. ■

Problem 4.3.5 : Write the function $x \vee y'$ in the disjunctive normal form in three variables x, y, z.

Solution: We have

$$x \vee y' = [x \wedge (y \vee y') \wedge (z \vee z')] \vee [y' \wedge (x \vee x') \wedge (z \vee z')]$$

$$= [\{(x \wedge y) \vee (x \wedge y')\} \wedge (z \vee z')] \vee [\{(y' \wedge x) \vee (y' \wedge x)\} \wedge (z \vee z')]$$

$$= (x \wedge y \wedge z) \wedge (x \wedge y \wedge z') \wedge (x \wedge y' \wedge z) \wedge (x \wedge y' \wedge z') \vee (y' \wedge x \wedge z) \vee$$
$$(y' \wedge x \wedge z') \wedge (y' \wedge x' \wedge z) \vee (y' \wedge x' \wedge z')$$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$$

which is disjunctive normal form of $x \vee y'$ in three variables $x, y, z.$ ■

Problem 4.3.6 : Find the Boolean expression that defines the function f given by

$f(0,0,0) = 0$ $\quad$ $f(1,0,0,) = 1$

$f(0,1,0) = 1$ $\quad$ $f(1,0,1) = 1$

$f(0,0,1) = 0$ $\quad$ $f(1,1,0) = 0$

$f(0,1,1) = 0$ $\quad$ $f(1,1,1) = 1$

Solution: We consider those values of $f(x,y,z)$ which are equal to 1.

The minterms corresponding to $f(0,1,0)$, $f(1,0,0)$, $f(1,0,1)$ and $f(1,1,1)$ will be $(x' \wedge y \wedge z')$, $(x \wedge y' \wedge z')$, $(x \wedge y' \wedge z)$ and $(x \wedge y \wedge z)$

Hence the function in DN form is,

$$f(x,y,z) = (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \wedge (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$$

Which can be simplified,

$$\begin{aligned} f(x,y,z) &= (x' \wedge y \wedge z') \vee x \wedge [(y' \wedge z') \vee (y' \wedge z) \vee (y \wedge z)] \\ &= (x' \wedge y \wedge z') \vee x \wedge [\{y' \wedge (z' \vee z)\} \vee (y \wedge z)] \\ &= (x' \wedge y \wedge z') \vee x \wedge [(y' \wedge 1) \vee (y \wedge z)] \\ &= (x' \wedge y \wedge z') \vee x \wedge [y' \vee (y \wedge z)] \\ &= (x' \wedge y \wedge z') \vee x \wedge [(y' \vee y) \wedge (y' \vee z)] \\ &= (x' \wedge y \wedge z') \vee x \wedge (y' \vee z) \\ &= (x' \wedge y \wedge z') \vee [(x \wedge y') \vee (x \wedge z)] \end{aligned}$$

, which is the required Boolean function. ■

Problem 4.3.7 : Find the Boolean (function) expression for the function f given by

$$f(x,y,z) = \begin{cases} 1 & when \quad x = z = 1, y = 0; x = 1, y = z = 0 \\ 0 & otherwise \end{cases}$$

Solution: We consider those values of $f(x,y,z)$ which are equal to 1. The minterms corresponding to $f(1,0,1)$, $f(1,0,0)$ will be $(x \wedge y' \wedge z) \& (x \wedge y' \wedge z')$

Hence the DN form of f is $= (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z')$. ■

Problem 4.3.8 : Let $f(x,y,z) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee z)$

be a Boolean expression over the two-valued Boolean algebra. Write $f(x,y,z)$ in disjunctive normal form.

Solution: We first find all values of $f(x,y,z)$ when x,y,z take values either 0 or 1.

$f(0,0,0)=(0\vee 0)\wedge(0\vee 1)\wedge(1\vee 0)=0\wedge 1\wedge 1=0$

$f(0,0,1)=(0\vee 0)\wedge(0\vee 0')\wedge(0'\vee 1)=0\wedge 1\wedge 1=0$

$f(0,1,0)=(0\vee 1)\wedge(0\vee 1')\wedge(0'\vee 0)=1\wedge 0\wedge 1=0$

$f(1,0,0)=(1\vee 0)\wedge(1\vee 0')\wedge(1'\vee 0)=1\wedge 1\wedge 0=0$

$f(0,1,1)=(0\vee 1)\wedge(0\vee 1')\wedge(0'\vee 1)=1\wedge 0\wedge 1=0$

$f(1,0,1)=(1\vee 0)\wedge(1\vee 0')\wedge(1'\vee 1)=1\wedge 1\wedge 1=1$

$f(1,1,0)==(1\vee 1)\wedge(1\vee 1')\wedge(1'\vee 0)=1\wedge 1\wedge 0=0$

$f(1,1,1)=(1\vee 1)\wedge(1\vee 1')\wedge(1'\vee 1)=1\wedge 1\wedge 1=1$

Now we consider those values of $f(x,y,z)$ which are equal to 1. The minterms corresponding to $f(1,0,1)$ and $f(1,1,1)$ will be $(x\wedge y'\wedge z)\&(x\wedge y\wedge z)$. Hence the Disjunctive normal form of f is

$$=(x\wedge y'\wedge z)\vee(x\wedge y\wedge z). \quad \blacksquare$$

4.4 Conjunctive Normal form :

In this section we discuss conjunctive normal form (CN form) which is dual of the DN form.

Definition (CN form) : A Boolean function f is said to be in conjuctive normal form (CN form) in n variables $x_1,x_2,...,....,x_n$ if f is meet of terms of the type $f_1(x_1)\vee f_2(x_2)\vee.....\vee f_n(x_n)$ where $f_i(x_i)=x_i$ or x_i', for all $i=1,2,........,n$ and no two terms are same.

Remark : A normal form is also called a canonical form.

Definition (Maxterm): A Boolean expression of n variables $x_1, x_2, ..., x_n$ is said to be a maxterm or maxterm polynomial if it is of the form,

$$f_1(x_1) \vee f_2(x_2) \vee f_3(x_3) \vee ...... \vee f_n(x_n)$$

Where $f_i(x_i) = x_i$ or x_i', for all $i = 1, 2, ..., n$.

Problem 4.4.1 : Put the function, $f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$ in the CN form.

Solution : Given,

$$f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$$

$$= [(x' \vee y'') \vee z'] \wedge (x'' \wedge z')$$

$$= (x' \vee y \vee z') \wedge (x \wedge z')$$

$$= (x' \vee y \vee z') \wedge [(x \wedge z') \vee (y \wedge y')]$$

$$= (x' \vee y \vee z') \wedge \{[(x \wedge z') \vee y] \wedge [(x \wedge z') \vee y']\}$$

$$= (x' \vee y \vee z') \wedge [(x \vee y) \wedge (z' \vee y) \wedge (x \vee y') \wedge (z' \vee y')]$$

$$= (x' \vee y \vee z') \wedge [\{(x \vee y) \vee (z \wedge z')\} \wedge \{(z' \vee y) \vee (x \wedge x')\}$$

$$\wedge \{(x \vee y') \vee (z \wedge z')\} \wedge \{(z' \vee y') \vee (x \wedge x')\}]$$

$$= (x' \vee y \vee z') \wedge (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (z' \vee y \vee x) \wedge (z' \vee y \vee x')$$

$$\wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (z' \vee y' \vee x) \wedge (x' \vee y' \vee z')$$

$$= (x' \vee y \vee z') \wedge (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z')$$

which is the required conjunctive normal form of f. ■

Problem 4.4.2 : Find the Boolean expression in CN form that defines the function f given by

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	0
0	1	0	1
1	0	0	0
0	1	1	0
1	0	1	0
1	1	0	0
1	1	1	1

Solution : We consider that values of $f(x,y,z)$ which are equal to 0. The maxterms corresponding to $f(0,0,1)$, $f(0,1,1)$, $f(1,0,0)$, $f(1,0,1)$, $f(1,1,0)$ will be

$(x'\vee y'\vee z)$, $(x'\vee y\vee z)$, $\left(x\vee y'\vee z'\right)$, $(x\vee y'\vee z)$, $(x\vee y\vee z')$

Hence the conjunctive normal form of f is

$$=(x'\vee y'\vee z)\wedge(x'\vee y\vee z)\wedge(x\vee y'\vee z')\wedge(x\vee y'\vee z)\wedge(x\vee y\vee z').\ \blacksquare$$

Problem 4.4.3 : Find the complement of the DN form

$$f = (x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z')$$

Solution : Given,

$$f = (x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z')$$

We know the complete DN form in 3-variables is,

$$(x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y' \wedge z)$$

$$\vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$$

Now, if we pickup the DN form f form the complete DN form then complement of f will contain the "left out" terms in the complete DN form

$$\therefore f' = (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z). \quad \blacksquare$$

Problem 4.4.4 : Find the CN form of the function $f = (x \wedge (y' \vee z)) \vee z'$ and then find its DN form from it.

Solution : Given

$$f = (x \wedge (y' \vee z)) \vee z'$$

$$= (x \vee z') \wedge ((y' \vee z) \vee z')$$

$$= (x \vee z') \wedge (y' \vee (z \vee z'))$$

$$= (x \vee z') \wedge (y' \vee 1)$$

$$= x \vee z' = (x \vee z') \vee (y' \wedge y)$$

$$= (x \vee z' \vee y) \wedge (x \vee z' \vee y')$$

$= (x \vee y \vee z') \wedge (x \vee y' \vee z'),$ which is the required CN form of f.

Now, we find the DN form $f = (x \vee y \vee z') \wedge (x \vee y' \vee z')$

We know, $f = (f')'$

$$= [\{(x \vee y \vee z') \wedge (x \vee y' \vee z')\}']'$$

$$= [(x \vee y \vee z)' \vee (x \vee y' \vee z')']'$$

$$= [(x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z)]'$$

We know the complete DN form in 3- variables is,

$(x \wedge y \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee$

$(x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z')$

Hence the DN form of f is

$(x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z')$. ■

Problem 4.4.5 : If $f = B_1 \vee B_2 \vee ...... \vee B_k$ be a Boolean function in n variables $x_1, x_2, ..., x_n$ in DN form where B_i are minterms then show that $B_1' \wedge B_2' \wedge ...... \wedge B_k'$ is the CN form of f'.

Solution: Given

$$f = B_1 \vee B_2 \vee ...... \vee B_k$$

$$\therefore f' = \left(B_1 \vee B_2 \vee ...... \vee B_k\right)'$$

$$= B_1' \wedge B_2' \wedge ...... \wedge B_k'$$

Here given each B_i being a minterm is of the form

$m_1 \wedge m_2 \wedge .... \wedge m_n$, where each $m_i = x_i$ or x_i', $i = 1, 2, ..., n.$

Thus $B_i' = m_1' \vee m_2' \vee .... \vee m_n'$, where each $m_i' = x_i$ or x_i', and therefore, B_i' is a maxterm.

Hence $B_1' \wedge B_2' \wedge ...... \wedge B_k'$ is the CN form of f'. ■

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